Complex Numbers Review Problems

1. The complex number z is defined by

$$z = 4\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right) + 4\sqrt{3}\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right).$$

(a) Express z in the form $re^{\mathrm{i}\theta}$, where r and θ have exact values.

(b) Find the cube roots of z, expressing in the form $re^{\mathrm{i}\theta}$, where r and θ have exact values.

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(Total 6 marks)

2. The polynomial $P(z) = z^3 + mz^2 + nz - 8$ is divisible by $(z + 1 + \mathrm{i})$, where $z \in \mathbb{C}$ and $m, n \in \mathbb{R}$. Find the value of m and of n.

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(Total 6 marks)

3. Let $u = 1 + \sqrt{3}\,\mathrm{i}$ and $v = 1 + \mathrm{i}$ where $\mathrm{i}^2 = -1$.

(a) (i) Show that $\dfrac{u}{v} = \dfrac{\sqrt{3}+1}{2} + \dfrac{\sqrt{3}-1}{2}\mathrm{i}$.

(ii) By expressing both u and v in modulus-argument form show that

$$\frac{u}{v} = \sqrt{2}\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right).$$

(iii) **Hence** find the **exact** value of $\tan\dfrac{\pi}{12}$ in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Z}$.

(15)

(b) Use mathematical induction to prove that for $n \in \mathbb{Z}^+$,

$$\left(1+\sqrt{3}\,\mathrm{i}\right)^n = 2^n\left(\cos\frac{n\pi}{3} + \mathrm{i}\sin\frac{n\pi}{3}\right).$$

(7)

(c) Let $z = \dfrac{\sqrt{2}v+u}{\sqrt{2}v-u}$.

Show that Re $z = 0$.

(6)
(Total 28 marks)

4. (a) Express the complex number $1+\mathrm{i}$ in the form $\sqrt{a}\mathrm{e}^{\mathrm{i}\frac{\pi}{b}}$, where $a, b \in \mathbb{Z}^+$.

(2)

(b) Using the result from (a), show that $\left(\dfrac{1+\mathrm{i}}{\sqrt{2}}\right)^n$, where $n \in \mathbb{Z}$, has only eight distinct values.

(5)

(c) **Hence** solve the equation $z^8 - 1 = 0$.

(2)
(Total 9 marks)

5. Find, in its simplest form, the argument of $(\sin\theta + \mathrm{i}(1-\cos\theta))^2$ where θ is an acute angle.

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(Total 7 marks)

6. Consider $w = \dfrac{z}{z^2+1}$ where $z = x + \mathrm{i}y$, $y \neq 0$ and $z^2 + 1 \neq 0$.

Given that Im $w = 0$, show that $|z| = 1$.

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(Total 7 marks)

7. (a) Use de Moivre's theorem to find the roots of the equation $z^4 = 1 - \mathrm{i}$.

(6)

(b) Draw these roots on an Argand diagram.

(2)

(c) If z_1 is the root in the first quadrant and z_2 is the root in the second quadrant, find $\dfrac{z_2}{z_1}$ in the form $a + \mathrm{i}b$.

(4)
(Total 12 marks)

8. Given that $(a + b\text{i})^2 = 3 + 4\text{i}$ obtain a pair of simultaneous equations involving a and b. Hence find the two square roots of $3 + 4\text{i}$.

(Total 7 marks)

9. Given that $|z| = \sqrt{10}$, solve the equation $5z + \dfrac{10}{z^*} = 6 - 18\text{i}$, where z^* is the conjugate of z.

(Total 7 marks)

10. Solve the simultaneous equations

$$\text{i}z_1 + 2z_2 = 3$$
$$z_1 + (1 - \text{i})z_2 = 4$$

giving z_1 and z_2 in the form $x + \text{i}y$, where x and y are real.

(Total 9 marks)

11. Find b where $\dfrac{2 + b\text{i}}{1 - b\text{i}} = \dfrac{7}{10} + \dfrac{9}{10}\text{i}$.

(Total 6 marks)

12. Given that $z = (b + \text{i})^2$, where b is real and positive, find the value of b when $\arg z = 60°$.

(Total 6 marks)

13. Consider the complex geometric series $\text{e}^{\text{i}\theta} + \frac{1}{2}\text{e}^{2\text{i}\theta} + \frac{1}{4}\text{e}^{3\text{i}\theta} + \ldots$

(a) Find an expression for z, the common ratio of this series.

(2)

(b) Show that $|z| < 1$.

(2)

(c) Write down an expression for the sum to infinity of this series.

(2)

(d) (i) Express your answer to part (c) in terms of $\sin\theta$ and $\cos\theta$.

(ii) Hence show that

$$\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \ldots = \frac{4\cos\theta - 2}{5 - 4\cos\theta}.$$

(10)

(Total 16 marks)

14. The roots of the equation $z^2 + 2z + 4 = 0$ are denoted by α and β?

(a) Find α and β in the form $r\text{e}^{\text{i}\theta}$.

(6)

(b) Given that α lies in the second quadrant of the Argand diagram, mark α and β on an Argand diagram.

(2)

(c) Use the principle of mathematical induction to prove De Moivre's theorem, which states that $\cos n\theta + \text{i}\sin n\theta = (\cos\theta + \text{i}\sin\theta)^n$ for $n \in \mathbb{Z}^+$.

(8)

(d) Using De Moivre's theorem find $\dfrac{\alpha^3}{\beta^2}$ in the form $a + \text{i}b$.

(4)

(e) Using De Moivre's theorem or otherwise, show that $\alpha^3 = \beta^3$.

(3)

(f) Find the exact value of $\alpha\beta^* + \beta\alpha^*$ where α^* is the conjugate of α and β^* is the conjugate of β.

(5)

(g) Find the set of values of n for which α^n is real.

(3)

(Total 31 marks)

Complex Numbers Review Problems - MarkScheme

1. (a) $z = 4\left(-\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\right)+4\sqrt{3}\left(\frac{\sqrt{3}}{2}+\frac{\mathrm{i}}{2}\right)$ A1A1

$= 8\left(\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}\right)=4+4\mathrm{i}\sqrt{3}$

$r = 8$ A1

$\theta=\frac{\pi}{3}$ or $60°$ A1

$\left(z=8\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{3}\right)}\right)$ N4

(b) $z^{\frac{1}{3}}=2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{9}\right)}\left(=2\mathrm{e}^{\mathrm{i}20°}\right)$ A1

$z^{\frac{1}{3}}=2\mathrm{e}^{\mathrm{i}\left(\frac{7\pi}{9}\right)}\left(=2\mathrm{e}^{\mathrm{i}140°}\right), z^{\frac{1}{3}}=2\mathrm{e}^{\mathrm{i}\left(\frac{13\pi}{9}\right)}\left(=2\mathrm{e}^{\mathrm{i}260°}\right)$ A1

Notes: *Do not allow any form other than* $re^{i\theta}$.
Both answers must be given for final A1.

[6]

2. **METHOD 1**

Using factor theorem (M1)
Substituting $z = -1 - \mathrm{i}$ into P(z) M1
$-(6 + n) + (2m - 2 - n)\mathrm{i} = 0$ A1
Equating both real and imaginary parts to zero M1
Hence $m = -2$ and $n = -6$ A1A1 N2

METHOD 2

Using Conjugate root theorem M1
Multiply $(z + 1 - \mathrm{i})(z + 1 + \mathrm{i}) = z^2 + 2z + 2$ M1
Let $P(z) = (z^2 + 2z + 2)(z - a)$ (M1)
$-2a = -8$ $a = 4$ A1
Hence $m = -2$ and $n = -6$ A1A1 N2

[6]

3. (a) (i) Using v^* where $v^* = 1 - \mathrm{i}$ (M1)

$\frac{u}{v}=\frac{(1+\sqrt{3}\mathrm{i})(1-\mathrm{i})}{(1+\mathrm{i})(1-\mathrm{i})}$ A1

$=\frac{1-\mathrm{i}+\sqrt{3}\mathrm{i}+\sqrt{3}}{2}$ A1A1

Note: *Award A1 for a correct numerator and A1 for a correct denominator.*

$\frac{u}{v}=\frac{\sqrt{3}+1}{2}+\frac{\sqrt{3}-1}{2}\mathrm{i}$ AG N0

(ii) $|u|=2$ and arg $u=\frac{\pi}{3}$ $\left(u=2\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)\right)$ A1A1

$|v|=2$ and arg $v=\frac{\pi}{4}$ $\left(v=\sqrt{2}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)\right)$ A1A1

$$\frac{u}{v}=\frac{2}{\sqrt{2}}\left(\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)+\mathrm{i}\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\right)$$ M1A1

$$\frac{u}{v}=\sqrt{2}\left(\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}\right)$$ AG N0

(iii) **METHOD 1**

Using arg $\frac{u}{v}$ to form $\frac{\pi}{12}=\arctan\frac{\frac{\sqrt{3}-1}{2}}{\frac{\sqrt{3}+1}{2}}$ (M1)(A1)

$$\tan\frac{\pi}{12}=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$ A1

$$=\frac{\left(\sqrt{3}-1\right)^2}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)}$$ M1

$= 2 - \sqrt{3}$ A1 N0

METHOD 2

$$\sqrt{2}\left(\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}\right)=\frac{\sqrt{3}+1}{2}+\frac{\sqrt{3}-1}{2}\mathrm{i}$$ (M1)

$\cos\frac{\pi}{12}=\frac{\sqrt{3}+1}{2\sqrt{2}}$ and $\sin\frac{\pi}{12}=\frac{\sqrt{3}-1}{2\sqrt{2}}$ (A1)

$$\tan\frac{\pi}{12}=\frac{\sqrt{3}-1}{\sqrt{3}+1}$$ A1

$$=\frac{\left(\sqrt{3}-1\right)^2}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)}$$ M1

$= 2 - \sqrt{3}$ A1 N0

Note: *Please check that $\sqrt{2}$ has been considered in either line 1 or line 2.*

(b) (Let P(n) be $(1+\sqrt{3}\mathrm{i})^n=2^n\left(\cos\frac{n\pi}{3}+\mathrm{i}\sin\frac{n\pi}{3}\right)$)

For $n = 1$: $2^1\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)=1+\sqrt{3}\mathrm{i}$, so P(1) is true A1

Assume P(k) is true, M1

$$\left(1+\sqrt{3}\mathrm{i}\right)^k=2^k\left(\cos\frac{k\pi}{3}+\mathrm{i}\sin\frac{k\pi}{3}\right)$$ (A1)

Consider P(k + 1)

$$\left(1+\sqrt{3}\mathrm{i}\right)^{k+1}=\left(1+\sqrt{3}\mathrm{i}\right)^k\left(1+\sqrt{3}\mathrm{i}\right)$$ M1

$$=2^k\left(\cos\frac{k\pi}{3}+\mathrm{i}\sin\frac{k\pi}{3}\right)2\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)$$ A1

$$=2^{k+1}\left(\cos\frac{(k+1)\pi}{3}+\mathrm{i}\sin\frac{(k+1)\pi}{3}\right)$$ A1

P(k) true implies P(k + 1) true, P(1) true so P(n) true $\forall\, n \in \mathbb{Z}^+$. R1 N0

(c) **METHOD 1**

$\sqrt{2}v + u = (\sqrt{2} + 1) + (\sqrt{2} + \sqrt{3})\,\mathrm{i}$ (M1)(A1)

$\sqrt{2}v - u = (\sqrt{2} - 1) + (\sqrt{2} - \sqrt{3})\,\mathrm{i}$ (A1)

$$\frac{\sqrt{2}v+u}{\sqrt{2}v-u} = \frac{(\sqrt{2}+1)+(\sqrt{2}+\sqrt{3})\mathrm{i}}{(\sqrt{2}-1)+(\sqrt{2}-\sqrt{3})\mathrm{i}} \times \frac{(\sqrt{2}-1)-(\sqrt{2}-\sqrt{3})\mathrm{i}}{(\sqrt{2}-1)-(\sqrt{2}-\sqrt{3})\mathrm{i}}$$ M1

$$\mathrm{Re}\left(\frac{\sqrt{2}v+u}{\sqrt{2}v-u}\right) = \frac{(\sqrt{2}+1)(\sqrt{2}-1)+(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})}{(\sqrt{2}-1)^2+(\sqrt{2}-\sqrt{3})^2}$$ (A1)

$$= \frac{2-1+(2-3)}{(\sqrt{2}-1)^2+(\sqrt{2}-\sqrt{3})^2}$$ A1

$= 0$ AG N0

Note: *If the candidate explains that to show that Re z = 0, it is only necessary to consider*
$$\left[(\sqrt{2}+1)+(\sqrt{2}+\sqrt{3}\,\mathrm{i})\right]\times\left[(\sqrt{2}-1)-(\sqrt{2}-\sqrt{3})\,\mathrm{i}\right]$$
then award as above.

METHOD 2

$$\sqrt{2}v+u = 2\left[\left(\cos\frac{\pi}{4}+\cos\frac{\pi}{3}\right)+\mathrm{i}\left(\sin\frac{\pi}{4}+\sin\frac{\pi}{3}\right)\right]$$ (M1)(A1)

$$\sqrt{2}v-u = 2\left[\left(\cos\frac{\pi}{4}-\cos\frac{\pi}{3}\right)+\mathrm{i}\left(\sin\frac{\pi}{4}-\sin\frac{\pi}{3}\right)\right]$$ (A1)

$$\frac{\sqrt{2}v+u}{\sqrt{2}v-u} = \frac{\left(\cos\frac{\pi}{4}+\cos\frac{\pi}{3}\right)+\mathrm{i}\left(\sin\frac{\pi}{4}+\sin\frac{\pi}{3}\right)}{\left(\cos\frac{\pi}{4}-\cos\frac{\pi}{3}\right)+\mathrm{i}\left(\sin\frac{\pi}{4}-\sin\frac{\pi}{3}\right)} \times \frac{\left(\cos\frac{\pi}{4}-\cos\frac{\pi}{3}\right)-\mathrm{i}\left(\sin\frac{\pi}{4}-\sin\frac{\pi}{3}\right)}{\left(\cos\frac{\pi}{4}-\cos\frac{\pi}{3}\right)-\mathrm{i}\left(\sin\frac{\pi}{4}-\sin\frac{\pi}{3}\right)}$$ M1

$$\mathrm{Re}\left(\frac{\sqrt{2}v+u}{\sqrt{2}v-u}\right) = \frac{\cos^2\frac{\pi}{4}-\cos^2\frac{\pi}{3}+\sin^2\frac{\pi}{4}-\sin^2\frac{\pi}{3}}{\left(\cos\frac{\pi}{4}-\cos\frac{\pi}{3}\right)^2+\left(\sin\frac{\pi}{4}-\sin\frac{\pi}{3}\right)^2}$$ (A1)

$$= \frac{\cos^2\frac{\pi}{4}-\sin^2\frac{\pi}{3}+\left(\cos^2\frac{\pi}{4}-\sin^2\frac{\pi}{3}\right)}{\left(\cos\frac{\pi}{4}-\cos\frac{\pi}{3}\right)^2+\left(\sin\frac{\pi}{4}-\sin\frac{\pi}{3}\right)^2}$$ A1

$= 0$ AG N0

Note: *If the candidate explains that to show that Re z = 0, it is only necessary to consider*
$$\left[\left(\cos\frac{\pi}{4}+\cos\frac{\pi}{3}\right)+\mathrm{i}\left(\sin\frac{\pi}{4}+\sin\frac{\pi}{3}\right)\right]\times\left[\left(\cos\frac{\pi}{4}-\cos\frac{\pi}{3}\right)-\mathrm{i}\left(\sin\frac{\pi}{4}-\sin\frac{\pi}{3}\right)\right]$$
then award as above.

METHOD 3

$$\frac{\sqrt{2}v+u}{\sqrt{2}v-u}=\frac{\sqrt{2}+\frac{u}{v}}{\sqrt{2}-\frac{u}{v}}$$ (M1)(A1)

$$=\frac{\left(\sqrt{2}+\frac{\sqrt{3}+1}{2}\right)-\frac{\sqrt{3}-1}{2}\mathrm{i}}{\left(\sqrt{2}-\frac{\sqrt{3}+1}{2}\right)-\frac{\sqrt{3}-1}{2}\mathrm{i}}$$ A1

$$=\frac{\left(\sqrt{2}+\frac{\sqrt{3}+1}{2}\right)+\frac{\sqrt{3}-1}{2}\mathrm{i}}{\left(\sqrt{2}-\frac{\sqrt{3}+1}{2}\right)-\frac{\sqrt{3}-1}{2}\mathrm{i}}\times\frac{\left(\sqrt{2}-\frac{\sqrt{3}+1}{2}\right)+\frac{\sqrt{3}-1}{2}\mathrm{i}}{\left(\sqrt{2}-\frac{\sqrt{3}+1}{2}\right)+\frac{\sqrt{3}-1}{2}\mathrm{i}}$$ M1

$$\mathrm{Re}\left(\frac{\sqrt{2}v+u}{\sqrt{2}v-u}\right)=2-\frac{\left(\sqrt{3}+1\right)^2}{4}-\frac{\left(\sqrt{3}-1\right)^2}{4}$$ A1

$$=2-\left(\frac{3+1+2\sqrt{3}+3+1-2\sqrt{3}}{4}\right)$$ A1

$$=0$$ AG N0

METHOD 4

$$\frac{\sqrt{2}v+u}{\sqrt{2}v-u}=\frac{\sqrt{2}+\frac{u}{v}}{\sqrt{2}-\frac{u}{v}}$$ (M1)(A1)

$$=\frac{\sqrt{2}+\sqrt{2}\left(\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}\right)}{\sqrt{2}-\sqrt{2}\left(\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}\right)}$$ A1

$$=\frac{\sqrt{2}\left(1+\cos\frac{\pi}{12}+\mathrm{i}\sin\frac{\pi}{12}\right)}{\sqrt{2}\left(1-\cos\frac{\pi}{12}-\mathrm{i}\sin\frac{\pi}{12}\right)}\times\frac{\left(1-\cos\frac{\pi}{12}\right)+\mathrm{i}\sin\frac{\pi}{12}}{\left(1-\cos\frac{\pi}{12}\right)+\mathrm{i}\sin\frac{\pi}{12}}$$ M1

$$\mathrm{Re}\left(\frac{\sqrt{2}v+u}{\sqrt{2}v-u}\right)=1-\cos^2\frac{\pi}{12}-\sin^2\frac{\pi}{12}$$ A1

$$=1-\left(\cos^2\frac{\pi}{12}-\sin^2\frac{\pi}{12}\right)$$ A1

$$=0$$ AG N0

[28]

4. (a) $1+\mathrm{i}=\sqrt{a}\mathrm{e}^{i\frac{\pi}{b}}$

$(\Rightarrow a=1^2+1^2=2)$

$\left(\tan\theta = \frac{1}{1} \Rightarrow \theta = \frac{\pi}{4}\right)$

$1 + \mathrm{i} = \sqrt{2}\mathrm{e}^{i\frac{\pi}{4}}$ A1A1 N2

(b) **EITHER**

$\left(\frac{1+\mathrm{i}}{\sqrt{2}}\right)^n = \left(\frac{\sqrt{2}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}}{\sqrt{2}}\right)^n = \mathrm{e}^{\frac{\mathrm{i}n\pi}{4}}$ M1A1

Let n = 0, 1, 2, 3, 4, 5, 6, 7 M1

Hence the eight distinct values are 1, $\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}, \mathrm{e}^{\frac{\mathrm{i}\pi}{2}}, \mathrm{e}^{\frac{\mathrm{i}3\pi}{4}}, \mathrm{e}^{\pi}, \mathrm{e}^{\frac{\mathrm{i}5\pi}{4}}, \mathrm{e}^{\frac{\mathrm{i}3\pi}{2}}, \mathrm{e}^{\frac{\mathrm{i}7\pi}{4}}$ A1

There are only eight distinct answers since the next answer would be $e^{\mathrm{i}2\pi}$ which is 1 and hence the arguments to all further answers would be the same as the first eight plus a multiple of 2π. R1

OR

$\left(\frac{1+\mathrm{i}}{\sqrt{2}}\right)^n = \left(\frac{\sqrt{2}\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}}{\sqrt{2}}\right)^n = \mathrm{e}^{\frac{\mathrm{i}n\pi}{4}}$ M1A1

Since $\mathrm{e}^{i(\theta+2\pi)} = e^{\mathrm{i}\theta}$ M1

$0 \le \frac{n\pi}{4} < 2\pi$ A1

Hence n can only take the values 0, 1, 2, 3, 4, 5, 6 and 7. R1

(c) From part (b) if we raise each of these roots to power 8 then the answer is 1. (M1)

Hence these are the eight roots to this equation.

$\Rightarrow z = 1, \mathrm{e}^{\frac{\mathrm{i}\pi}{4}}, \mathrm{e}^{\frac{\mathrm{i}\pi}{2}}, \mathrm{e}^{\frac{\mathrm{i}3\pi}{4}}, \mathrm{e}^{\pi}, \mathrm{e}^{\frac{\mathrm{i}5\pi}{4}}, \mathrm{e}^{\frac{\mathrm{i}3\pi}{2}}, \mathrm{e}^{\frac{\mathrm{i}7\pi}{4}}$ A1

[9]

5. $(\sin\theta + \mathrm{i}(1-\cos\theta))^2 = \sin^2\theta - (1-\cos\theta)^2 + \mathrm{i}\,2\sin\theta(1-\cos\theta)$ M1A1

Let α be the required argument.

$\tan\alpha = \frac{2\sin\theta(1-\cos\theta)}{\sin^2\theta - (1-\cos\theta)^2}$ M1

$= \frac{2\sin\theta(1-\cos\theta)}{(1-\cos^2\theta)-(1-2\cos\theta+\cos^2\theta)}$ (M1)

$= \frac{2\sin\theta(1-\cos\theta)}{2\cos\theta(1-\cos\theta)}$ A1

$= \tan\theta$ A1

$\alpha = \theta$ A1

[7]

6. METHOD 1

Substituting $z = x + \mathrm{i}y$ to obtain $w = \frac{x+y\mathrm{i}}{(x+y\mathrm{i})^2+1}$ (A1)

$w = \frac{x+y\mathrm{i}}{x^2-y^2+1+2xy\mathrm{i}}$ A1

Use of $(x^2 - y^2 + 1 - 2xy\text{i})$ to make the denominator real. M1

$= \dfrac{(x+y\text{i})(x^2-y^2+1-2xy\text{i})}{(x^2-y^2+1)^2+4x^2y^2}$ A1

$\text{Im } w = \dfrac{y(x^2-y^2+1)-2x^2y}{(x^2-y^2+1)^2+4x^2y^2}$ (A1)

$= \dfrac{y(1-x^2-y^2)}{(x^2-y^2+1)^2+4x^2y^2}$ A1

$\text{Im } w = 0 \Rightarrow 1 - x^2 - y^2 = 0$ *ie* $|z| = 1$ as $y \neq 0$ R1AG N0

METHOD 2

$w(z^2 + 1) = z$ (A1)

$w(x^2 - y^2 + 1 + 2\text{i}xy) = x + y\text{i}$ A1

Equating real and imaginary parts

$w(x^2 - y^2 + 1) = x$ and $2wx = 1$, $y \neq 0$ M1A1

Substituting $w = \dfrac{1}{2x}$ to give $\dfrac{x}{2} - \dfrac{y^2}{2x} + \dfrac{1}{2x} = x$ A1

$-\dfrac{1}{2x}(y^2 - 1) = \dfrac{x}{2}$ or equivalent (A1)

$x^2 + y^2 = 1$, *ie* $z = 1$ as $y \neq 0$ R1AG

[7]

7. (a) $z = (1-\text{i})^{\frac{1}{4}}$

Let $1 - \text{i} = r(\cos\theta + \text{i}\sin\theta)$

$\Rightarrow r = \sqrt{2}$ A1

$\theta = -\dfrac{\pi}{4}$ A1

$z = \left(\sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right) + \text{i}\sin\left(-\dfrac{\pi}{4}\right)\right)\right)^{\frac{1}{4}}$ M1

$= \left(\sqrt{2}\left(\cos\left(-\dfrac{\pi}{4} + 2n\pi\right) + \text{i}\sin\left(-\dfrac{\pi}{4} + 2n\pi\right)\right)\right)^{\frac{1}{4}}$

$= 2^{\frac{1}{8}}\left(\cos\left(-\dfrac{\pi}{16} + \dfrac{n\pi}{2}\right) + \text{i}\sin\left(-\dfrac{\pi}{16} + \dfrac{n\pi}{2}\right)\right)$ M1

$= 2^{\frac{1}{8}}\left(\cos\left(-\dfrac{\pi}{16}\right) + \text{i}\sin\left(-\dfrac{\pi}{16}\right)\right)$

Note: Award M1 above for this line if the candidate has forgotten to add 2π and no other solution given.

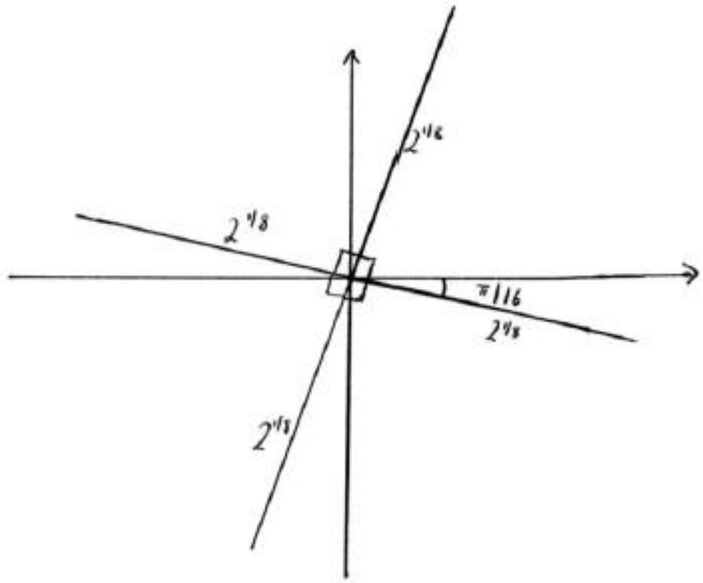
$= 2^{\frac{1}{8}}\left(\cos\left(\dfrac{7\pi}{16}\right) + \text{i}\sin\left(\dfrac{7\pi}{16}\right)\right)$

$= 2^{\frac{1}{8}}\left(\cos\left(\dfrac{15\pi}{16}\right) + \text{i}\sin\left(\dfrac{15\pi}{16}\right)\right)$

$= 2^{\frac{1}{8}}\left(\cos\left(-\frac{9\pi}{16}\right)+\mathrm{i}\sin\left(-\frac{9\pi}{16}\right)\right)$ A2

Note: Award A1 for 2 correct answers. Accept any equivalent form.

(b)



A2

Note: Award A1 for roots being shown equidistant from the origin and one in each quadrant.
A1 for correct angular positions. It is not necessary to see written evidence of angle, but must agree with the diagram.

(c) $\frac{z_2}{z_1} = \frac{2^{\frac{1}{8}}\left(\left(\cos\frac{15\pi}{16}\right)+\mathrm{i}\sin\left(\frac{15\pi}{16}\right)\right)}{2^{\frac{1}{8}}\left(\left(\cos\frac{7\pi}{16}\right)+\mathrm{i}\sin\left(\frac{7\pi}{16}\right)\right)}$ M1A1

$= \cos\frac{\pi}{2}+\mathrm{i}\sin\frac{\pi}{2}$ (A1)

$= \mathrm{i}$ A1 N2

$(\Rightarrow a = 0, b = 1)$

[12]

8. $a^2 + 2\mathrm{i}ab - b^2 = 3 + 4\mathrm{i}$

Equate real and imaginary parts (M1)

$a^2 - b^2 = 3$, $2ab = 4$ A1

Since $b = \frac{2}{a}$

$\Rightarrow a^2 - \frac{4}{a^2} = 3$ (M1)

$\Rightarrow a^4 - 3a^2 - 4 = 0$ A1

Using factorisation or the quadratic formula (M1)

$\Rightarrow a = \pm 2$

$\Rightarrow b = \pm 1$

$\Rightarrow \sqrt{3+4\mathrm{i}} = 2 + \mathrm{i}, -2 - \mathrm{i}$ A1A1

[7]

9. $5zz^* + 10 = (6 - 18\mathrm{i})z^*$ M1

Let $z = a + \mathrm{i}b$

$5 \times 10 + 10 = (6 - 18\mathrm{i})(a - b\mathrm{i})$ $(= 6a - 6b\mathrm{i} - 18a\mathrm{i} - 18b)$ M1A1

Equate real and imaginary parts (M1)

$\Rightarrow 6a - 18b = 60$ and $6b + 18a = 0$

$\Rightarrow a = 1$ and $b = -3$ A1A1

$z = 1 - 3\mathrm{i}$ A1

[7]

10. $\mathrm{i}z_1 + 2z_2 = 3 \Rightarrow z_2 = -\frac{1}{2}\mathrm{i}z_1 + \frac{3}{2}$

$z_1 + (1 - \mathrm{i})z_2 = 4$

$\Rightarrow z_1 + (1 - \mathrm{i})\left(-\frac{1}{2}\mathrm{i}z_1 + \frac{3}{2}\right) = 4$ M1A1

$\Rightarrow z_1 - \frac{1}{2}\mathrm{i}z_1 + \frac{3}{2} + \frac{1}{2}\mathrm{i}^2 z_1 - \frac{3}{2}\mathrm{i} = 4$

$\Rightarrow \frac{1}{2}z_1 - \frac{1}{2}\mathrm{i}z_1 = \frac{5}{2} + \frac{3}{2}\mathrm{i}$

$\Rightarrow z_1 - \mathrm{i}z_1 = 5 + 3\mathrm{i}$ A1

EITHER

Let $z_1 = x + \mathrm{i}y$ (M1)

$\Rightarrow x + \mathrm{i}y - \mathrm{i}x - \mathrm{i}^2 y = 5 + 3\mathrm{i}$

Equate real and imaginary parts M1

$\Rightarrow x + y = 5$

$\underline{-x + y = 3}$

$2y = 8$

$y = 4 \Rightarrow x = 1$ i.e. $z_1 = 1 + 4\mathrm{i}$ A1A1

$z_2 = -\frac{1}{2}\mathrm{i}(1 + 4\mathrm{i}) + \frac{3}{2}$ M1

$z_2 = -\frac{1}{2}\mathrm{i} - 2\mathrm{i}^2 + \frac{3}{2}$

$z_2 = \frac{7}{2} - \frac{1}{2}\mathrm{i}$ A1

OR

$z_1 = \frac{5 + 3\mathrm{i}}{1 - \mathrm{i}}$ M1

$z_1 = \frac{(5 + 3\mathrm{i})(1 + \mathrm{i})}{(1 - \mathrm{i})(1 + \mathrm{i})}$ $\left(= \frac{5 + 8\mathrm{i} - 3}{2}\right)$ M1A1

$z_1 = 1 + 4\mathrm{i}$ A1

$z_2 = -\frac{1}{2}\mathrm{i}(1 + 4\mathrm{i}) + \frac{3}{2}$ M1

$z_2 = -\frac{1}{2}\mathrm{i} - 2\mathrm{i}^2 + \frac{3}{2}$

$z_2 = \frac{7}{2} - \frac{1}{2}\mathrm{i}$ A1

[9]

11. **METHOD 1**

$20 + 10b\mathrm{i} = (1 - b\mathrm{i})(-7 + 9\mathrm{i})$ (M1)

$20 + 10b\mathrm{i} = (-7 + 9b) + (9 + 7b)\mathrm{i}$ A1A1

Equate real and imaginary parts (M1)

EITHER

$-7 + 9b = 20$

$b = 3$ (M1)A1

OR

$10b = 9 + 7b$

$3b = 9$

$b = 3$ (M1)A1

METHOD 2

$= \frac{(2+bi)(1+bi)}{(1-bi)(1+bi)} = \frac{-7+9i}{10}$ (M1)

$\frac{2-b^2+3bi}{1+b^2} = \frac{-7+9i}{10}$ A1

Equate real and imaginary parts (M1)

$\frac{2-b^2}{1+b^2} = -\frac{7}{10}$ Equation A

$\frac{3b}{1+b^2} = \frac{9}{10}$ Equation B

From equation A

$20 - 10b^2 = -7 - 7b^2$

$3b^2 = 27$

$b = \pm 3$ A1

From equation B

$30b = 9 + 9b^2$

$3b^2 - 10b + 3 = 0$

By factorisation or using the quadratic formula

$b = \frac{1}{3}$ or 3 A1

Since 3 is the common solution to both equations $b = 3$ R1

[6]

12. **METHOD 1**

since $b > 0$ (M1)

$\Rightarrow \arg(b + i) = 30°$ A1

$\frac{1}{b} = \tan 30°$ M1A1

$b = \sqrt{3}$ A2 N2

METHOD 2

$\arg(b + i)^2 = 60° \Rightarrow \arg(b^2 - 1 + 2bi) = 60°$ M1

$\frac{2b}{(b^2-1)} = \tan 60° = \sqrt{3}$ M1A1

$\sqrt{3}b^2 - 2b - \sqrt{3} = 0$ A1

$(\sqrt{3}b+1)(b-\sqrt{3}) = 0$

since $b > 0$ (M1)

$b = \sqrt{3}$ A1 N2

[6]

13. (a) $z = \frac{\frac{1}{2}e^{2i\theta}}{e^{i\theta}}$ (M1)

$z = \frac{1}{2}e^{i\theta}$ A1N2

(b) $|z| = \frac{1}{2}$ A2

$|z| < 1$ AG

(c) Using $S_\infty = \frac{a}{1-r}$ (M1)

$S_\infty = \frac{e^{i\theta}}{1-\frac{1}{2}e^{i\theta}}$ A1 N2

(d) (i) $S_\infty = \frac{e^{i\theta}}{1-\frac{1}{2}e^{i\theta}} = \frac{\text{cis}\,\theta}{1-\frac{1}{2}\text{cis}\,\theta}$ (M1)

$\frac{\cos\theta + i\sin\theta}{1-\frac{1}{2}(\cos\theta + i\sin\theta)}$ (A1)

Also $S_\infty = e^{i\theta} + \frac{1}{2}e^{2i\theta} + \frac{1}{4}e^{3i\theta} + \ldots$

$= \text{cis}\,\theta + \frac{1}{2}\text{cis}2\theta + \frac{1}{4}\text{cis}3\theta + \ldots$ (M1)

$S_\infty = \left(\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \ldots\right) + i\left(\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{4}\sin 3\theta + \ldots\right)$ A1

(ii) Taking real parts,

$\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \ldots = \text{Re}\left(\frac{\cos\theta + i\sin\theta}{1-\frac{1}{2}(\cos\theta + i\sin\theta)}\right)$ A1

$= \text{Re}\left(\frac{(\cos\theta + i\sin\theta)}{\left(1-\frac{1}{2}\cos\theta - \frac{1}{2}i\sin\theta\right)} \times \frac{1-\frac{1}{2}\cos\theta + \frac{1}{2}i\sin\theta}{\left(1-\frac{1}{2}\cos\theta + \frac{1}{2}i\sin\theta\right)}\right)$ M1

$= \frac{\cos\theta - \frac{1}{2}\cos^2\theta - \frac{1}{2}\sin^2\theta}{\left(1-\frac{1}{2}\cos\theta\right)^2 + \frac{1}{4}\sin^2\theta}$ A1

$= \frac{\left(\cos\theta - \frac{1}{2}\right)}{1 - \cos\theta + \frac{1}{4}(\sin^2\theta + \cos^2\theta)}$ A1

$= \frac{(2\cos\theta - 1) \div 2}{(4 - 4\cos\theta + 1) \div 4} = \frac{4(2\cos\theta - 1)}{2(5 - 4\cos\theta)}$ A1

$= \frac{4\cos\theta - 2}{5 - 4\cos\theta}$ A1AG N0

[25]

14. (a) $z = \dfrac{-2 \pm \sqrt{4-16}}{2} = -1 \pm i\sqrt{3}$ M1

$-1 + i\sqrt{3} = re^{i\theta} \Rightarrow r = 2$ A1

$\theta = \arctan \dfrac{\sqrt{3}}{-1} = \dfrac{2\pi}{3}$ A1

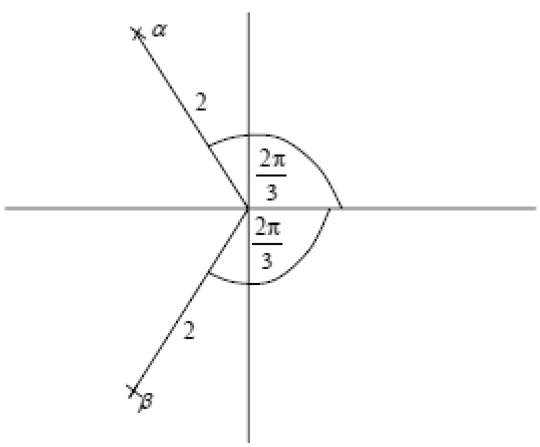
$-1 - i\sqrt{3} = re^{i\theta} \Rightarrow r = 2$

$\theta = \arctan \dfrac{\sqrt{3}}{-1} = -\dfrac{2\pi}{3}$ A1

$\Rightarrow \alpha = 2e^{i\frac{2\pi}{3}}$ A1

$\Rightarrow \beta = 2e^{-i\frac{2\pi}{3}}$ A1

(b)



A1A1

(c) $\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n$

Let $n = 1$

Left hand side $= \cos 1\theta + i \sin 1\theta = \cos \theta + i \sin \theta$

Right hand side $= (\cos \theta + i \sin \theta)^1 = \cos \theta + i \sin \theta$

Hence true for $n = 1$ M1A1

Assume true for $n = k$ M1

$\cos k\theta + i \sin k\theta = (\cos \theta + i \sin \theta)^k$

$\Rightarrow \cos(k+1)\theta + i \sin(k+1)\theta = (\cos\theta + i \sin \theta)^k(\cos \theta + i \sin \theta)$ M1A1

$= (\cos k\theta + i \sin k\theta)(\cos \theta + i \sin \theta)$

$= \cos k\theta \cos\theta - \sin k\theta \sin \theta + i(\cos k\theta \sin\theta + \sin k\theta \cos\theta)$ A1

$= \cos(k+1)\theta + i \sin(k+1)\theta$ A1

Hence if true for $n = k$, true for $n = k + 1$

However if it is true for $n = 1$

$\Rightarrow$ true for $n = 2$ *etc.* R1

$\Rightarrow$ hence proved by induction

(d) $\dfrac{\alpha^3}{\beta^2} = \dfrac{8e^{i2\pi}}{4e^{-i\frac{4\pi}{3}}} = 2e^{i\frac{4\pi}{3}}$ A1

$= 2\cos\dfrac{4\pi}{3} + 2i\sin\dfrac{4\pi}{3}$ (M1)

$= -\dfrac{2}{2} - 2\dfrac{i\sqrt{3}}{2} = -1 - i\sqrt{3}$ A1A1

(e) $a^3 = 8e^{i2\pi}$ A1

$\beta^3 = 8e^{-i2\pi}$ A1

Since $e^{2\pi}$ and $e^{-2\pi}$ are the same $\alpha^3 = \beta^3$ R1

(f) **EITHER**

$\alpha = -1 + i\sqrt{3}$ $\beta = -1 - i\sqrt{3}$

$\alpha^* = -1 - i\sqrt{3}$ $\beta^* = -1 + i\sqrt{3}$ A1

$\alpha\beta^* = (-1 + i\sqrt{3})(-1 + i\sqrt{3}) = 1 - 2i\sqrt{3} - 3 = 2 - 2i\sqrt{3}$ M1A1

$\beta\alpha^* = (-1 - i\sqrt{3})(-1 - i\sqrt{3}) = 1 + 2i\sqrt{3} - 3 = -2 + 2i\sqrt{3}$ A1

$\Rightarrow \alpha\beta^* + \beta\alpha^* = -4$ A1

OR

Since $\alpha^* = \beta$ and $\beta^* = \alpha$

$\alpha\beta^* = 2e^{i\frac{2\pi}{3}} \times 2e^{i\frac{2\pi}{3}} = 4e^{i\frac{4\pi}{3}}$ M1A1

$\beta\alpha^* = 2e^{-i\frac{2\pi}{3}} \times 2e^{-i\frac{2\pi}{3}} = 4e^{-i\frac{4\pi}{3}}$ A1

$\alpha\beta^* + \beta\alpha^* = 4\left(e^{i\frac{4\pi}{3}} + e^{-i\frac{4\pi}{3}}\right)$

$= 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} + \cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3}\right)$ A1

$= 8\cos\frac{4\pi}{3} = 8 \times -\frac{1}{2} = -4$ A1

(g) $\alpha^n = 2^n e^{i2\frac{\pi n}{3}}$ M1A1

This is real when n is a multiple of 3 R1

i.e. $n = 3N$ where $N \in \mathbb{Z}^+$

[31]