December 29, 2018

The jungle
HW
- Spread it out
- The problem is irrelevant!

Return Exams - take a look
Test corrections as a take home, Homework Quiz, Due Mon, 1/14

GoodWorkShown.pdf

Problem solving techniques:
BREAK THE PROBLEM DOWN - product rule
Write down the steps - make a plan
Communicate to others
Use mathematical notation correctly - forces you to think
What do I need? What do I have? How are they connected? What step can I take?

Coming up: **Ch 17 Quiz - Wed, 1/9**
- Kinematics
- Areas under curves & integration
- Applications of integration

--------- President's Day --------
- Discrete Random Variables

--------- Spring Break --------
- Continuous Random Variables

Chapter 17
Applications of differential calculus

A Kinematics
B Rates of change
C Optimisation

Chapter 17 Applications of differential calculus

A KINEMATICS

Objectives
1. Use derivatives to solve problems involving motion.

17A.1: #2-4 (Kinematics)
17A.2: #6-9 (Velocity & Acceleration)
17B: #1-16 by 3 (Rates of change)
QB #3,4,12
Test Corrections (Due Monday, 1/14)

Consider the height of a ball thrown up from the top of a 60 foot building at an initial speed of 12 feet per second. The ball's height can be modeled by:

$$h(t) = -16t^2 + 12t + 60$$

The change of position over time is also known as ***velocity***. $\text{Velocity} = \frac{\Delta\text{Position}}{\Delta\text{Time}}$

Using derivatives, we can find instantaneous velocity by letting $\Delta\text{Time} \to 0$

$$\text{Instantaneous Velocity} = \lim_{\Delta Time \to 0} \frac{\Delta Position}{\Delta Time} = \frac{d(\text{Position})}{d(\text{Time})} = \frac{dh}{dt} = h'(t)$$

In the example, velocity is given by $v(t) = -32t + 12$. Notice that when $t = 0$, the velocity is 12 ft/sec (the initial speed!) and that the velocity increases (in a downward direction) 32 ft/sec every second!

That ***change in velocity with respect to time*** is called ***acceleration***. It is also the ***second derivative*** of position (in this case height) with respect to time.

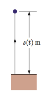
$$\text{Instantaneous Acceleration} = \lim_{\Delta Time \to 0} \frac{\Delta Velocity}{\Delta Time} = \frac{d(\text{Velocity})}{d(\text{Time})} = \frac{d^2h}{dt^2} = h''(t)$$

In the above example, the acceleration is simply, -32 ft/sec^2 - gravity!

Notice the positions of the superscripts in Leibnitz notation - the reasoning for this will become apparent later. It is read "dee-two h dee-tee squared"

Recall that $f''(x)$ or $f^{(2)}(x)$ is the ***second derivative*** of f with respect to x. It represents the "slope of the slope" or the ***curvature*** of f. In a motion context, $f''(t)$ is acceleration

A stone is projected vertically so that its position above ground level after t seconds is given by $s(t) = 98t - 4.9t^2$ metres, $t \geqslant 0$.



a Find the velocity and acceleration functions for the stone and draw sign diagrams for each function.
b Find the initial position and velocity of the stone.
c Describe the stone's motion at times $t = 5$ and $t = 12$ seconds.
d Find the maximum height reached by the stone.
e Find the time taken for the stone to hit the ground.

a) v(t) = 98 – 9.8t   a(t) = –9.8
b) s(0) = 0 m above ground;   v(0) = 98 m/sec upward
c) s(5) = 368 ft above ground;   v(5) = 49 m/sec upward   a(5) = –9.8 m/sec^2 slowing
s(12) = 470 ft above ground;   v(12) = –19.6 m/sec (down)   a(5) = –9.8 m/sec^2 speeding up
d) v(t) = 98 – 9.8t = 0 when t = 10 sec. and v changes sign at t = 10 ∴ max height is s(10) = 490 ft above ground
e) s(t) = 0 when 98t – 4.9t^2 = 0 or 4.9t(20 – t) = 0 so stone hits at t = 20. Symmetry also works since s(0) = 0

The values of position, velocity, and acceleration at $t = 0$ are called the ***initial conditions***.

Another common context is motion along a straight line. Consider the location, velocity, and acceleration of an object on the end of a spring for example.

Position
- A given problem involves an ***origin.*** It is generally the position of the object at $t = 0$, (not always). Know where it is in a given problem! In the spring problem, it can be the position of the spring at rest or it can be the location where the spring is attached to some support.
- Coordinate systems are intuitive or given. Up and right are > 0 left and down < 0.
- ***Displacement*** (often we use the letter s) is ***the signed distance from the origin.*** It is a ***vector*** as it has magnitude and direction relative to the origin. Note the distinction between displacement and distance travelled.
- ***Position*** can be considered as a point (scalar) or as a displacement from 0 (vector). Watch carefully when this word is used.

Velocity
- ***Average velocity*** is the net change in position divided by elapsed time.

$$\frac{\Delta\text{position}}{\Delta\text{time}} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

Note that ***velocity*** is also a ***vector*** as it has direction and magnitude relative to the origin. ***Speed***, on the other hand, is a ***scalar*** - the magnitude of velocity.

- ***Instantaneous velocity*** is the instantaneous rate of change of displacement vs time, better known as:

$$v(t) = \frac{ds}{dt} = s'(t)$$

Acceleration
- ***Average acceleration*** is the net change in velocity divided by elapsed time.

$$\frac{\Delta\text{velocity}}{\Delta\text{time}} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

Note that acceleration is also a ***vector*** as it has direction and magnitude relative to the origin.

- ***Instantaneous acceleration*** is the instantaneous rate of change of velocity vs time, better known as:

$$\lim_{\Delta time \to 0} \frac{\Delta velocity}{\Delta time} = \frac{dv}{dt} = v'(t) = \frac{d^2s}{dt^2} = s''(t)$$

Notice that an object is speeding up when the signs of the velocity and acceleration are the same because the acceleration is ***in the same direction*** as the motion. It is slowing down when the signs of the velocity and acceleration are different because the acceleration ***opposes*** the direction of motion.

A particle moves in a straight line with displacement from O given by $s(t) = 3t - t^2$ metres at time t seconds. Find:

a the average velocity for the time interval from $t = 2$ to $t = 5$ seconds
b the average velocity for the time interval from $t = 2$ to $t = 2 + h$ seconds
c $\lim_{h \to 0} \frac{s(2+h) - s(2)}{h}$ and comment on its significance.

a average velocity
$$= \frac{s(5) - s(2)}{5 - 2} = \frac{(15 - 25) - (6 - 4)}{3} = \frac{-10 - 2}{3} = -4 \text{ m s}^{-1}$$

b average velocity
$$= \frac{s(2+h) - s(2)}{2 + h - 2} = \frac{3(2+h) - (2+h)^2 - 2}{h} = \frac{6 + 3h - 4 - 4h - h^2 - 2}{h} = \frac{-h - h^2}{h} = -1 - h \text{ m s}^{-1} \text{ provided } h \neq 0$$

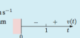
c
$$\lim_{h \to 0} \frac{s(2+h) - s(2)}{h} = \lim_{h \to 0} (-1 - h) \quad \{\text{since } h \neq 0\} = -1 \text{ m s}^{-1}$$

This is the instantaneous velocity of the particle at time $t = 2$ seconds.

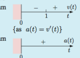
A particle moves in a straight line with position relative to O given by $s(t) = t^3 - 3t + 1$ cm, where t is the time in seconds, $t \geqslant 0$.

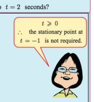
a Find expressions for the particle's velocity and acceleration, and draw sign diagrams for each of them.
b Find the initial conditions and hence describe the motion at this instant.
c Describe the motion of the particle at $t = 2$ seconds.
d Find the position of the particle when changes in direction occur.
e Draw a motion diagram for the particle.
f For what time interval is the particle's speed increasing?
g What is the total distance travelled in the time from $t = 0$ to $t = 2$ seconds?

a $s(t) = t^3 - 3t + 1$ cm
$\therefore\ v(t) = 3t^2 - 3$ $\{\text{as } v(t) = s'(t)\}$
$= 3(t^2 - 1)$
$= 3(t+1)(t-1)$ cm s^{-1}
which has sign diagram



and $a(t) = 6t$ cm s^{-2} $\{\text{as } a(t) = v'(t)\}$
which has sign diagram



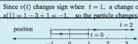


b When $t = 0$, $s(0) = 1$ cm
$v(0) = -3$ cm s^{-1}
$a(0) = 0$ cm s^{-2}
$\therefore$ the particle is 1 cm to the right of O, moving to the left at a speed of 3 cm s^{-1}.

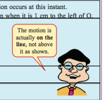
c When $t = 2$, $s(2) = 8 - 6 + 1 = 3$ cm
$v(2) = 12 - 3 = 9$ cm s^{-1}
$a(2) = 12$ cm s^{-2}
$\therefore$ the particle is 3 cm to the right of O, moving to the right at a speed of 9 cm s^{-1}.
Since a and v have the same sign, the speed of the particle is increasing.

d Since $v(t)$ changes sign when $t = 1$, a change of direction occurs at this instant.
$s(1) = 1 - 3 + 1 = -1$, so the particle changes direction when it is 1 cm to the left of O.

e



As $t \to \infty$, $s(t) \to \infty$ and $v(t) \to \infty$.



f Speed is increasing when $v(t)$ and $a(t)$ have the same sign. This is for $t \geqslant 1$.

g Total distance travelled $= 2 + 4 = 6$ cm.

B RATES OF CHANGE

Objectives

1. Use derivatives to solve problems involving rates of change.
2. Use derivatives to find time when motion reverses direction.

Since a derivative function represents the rate of change of a given function, derivatives can be used for many problems involving rates of change.

According to a psychologist, the ability of a person to understand spatial concepts is given by $A = \frac{1}{3}\sqrt{t}$ where t is the age in years, $5 \leqslant t \leqslant 18$.

a Find the rate of improvement in ability to understand spatial concepts when a person is:
  i 9 years old **ii** 16 years old.

b Explain why $\frac{dA}{dt} > 0$ for $5 \leqslant t \leqslant 18$. Comment on the significance of this result.

c Explain why $\frac{d^2A}{dt^2} < 0$ for $5 \leqslant t \leqslant 18$. Comment on the significance of this result.

a $A = \frac{1}{3}\sqrt{t} = \frac{1}{3}t^{\frac{1}{2}}$ $\quad \therefore \frac{dA}{dt} = \frac{1}{6}t^{-\frac{1}{2}} = \frac{1}{6\sqrt{t}}$

i When $t = 9$, $\frac{dA}{dt} = \frac{1}{18}$
$\therefore$ the rate of improvement is $\frac{1}{18}$ units per year for a 9 year old.

ii When $t = 16$, $\frac{dA}{dt} = \frac{1}{24}$
$\therefore$ the rate of improvement is $\frac{1}{24}$ units per year for a 16 year old.

b Since $\sqrt{t}$ is never negative, $\frac{1}{6\sqrt{t}}$ is never negative

$\therefore \frac{dA}{dt} > 0$ for all $5 \leqslant t \leqslant 18$.

This means that the ability to understand spatial concepts increases with age.

c $\frac{dA}{dt} = \frac{1}{6}t^{-\frac{1}{2}}$

$\therefore \frac{d^2A}{dt^2} = -\frac{1}{12}t^{-\frac{3}{2}} = -\frac{1}{12t\sqrt{t}}$

$\therefore \frac{d^2A}{dt^2} < 0$ for all $5 \leqslant t \leqslant 18$.

This means that while the ability to understand spatial concepts increases with time, the rate of increase slows down with age.

The cost in dollars of producing x items in a factory each day is given by

$$C(x) = \underbrace{0.000\,13x^3 + 0.002x^2}_{\text{labour}} + \underbrace{5x}_{\text{raw materials}} + \underbrace{2200}_{\text{fixed costs}}$$

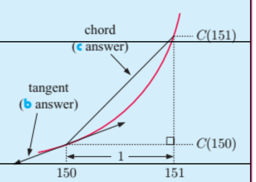
a Find $C'(x)$, which is called the marginal cost function.

b Find the marginal cost when 150 items are produced. Interpret this result.

c Find $C(151) - C(150)$. Compare this with the answer in **b**.

a The marginal cost function is
$C'(x) = 0.000\,39x^2 + 0.004x + 5$ dollars per item.

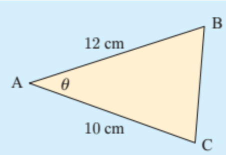
b $C'(150) = \$14.38$
This is the rate at which the costs are increasing with respect to the production level x when 150 items are made per day.
It gives an estimate of the cost increase in total cost from making the 151st item.



c $C(151) - C(150) \approx \$3448.19 - \$3433.75$
$\approx \$14.44$

This is the actual increase in total cost from making the 151st item each day, so the answer in **b** gives a good estimate.

Find the rate of change in the area of triangle ABC as θ changes, at the time when $\theta = 60°$.



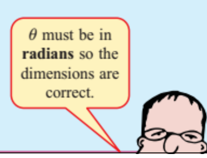
Area $A = \frac{1}{2} \times 10 \times 12 \times \sin\theta$ $\quad \{\text{Area} = \frac{1}{2}bc\sin A\}$

$\therefore A = 60\sin\theta$ cm^2

$\therefore \frac{dA}{d\theta} = 60\cos\theta$

When $\theta = \frac{\pi}{3}$, $\cos\theta = \frac{1}{2}$

$\therefore \frac{dA}{d\theta} = 30$ cm^2 per radian



Yes, this is brutal.
Spread it out!
If you can't finish, do some of each!

17A.1: #2-4 (Kinematics)
17A.2: #6-9 (Velocity & Acceleration)
17B: #1-16 by 3 (Rates of change)
QB #3,4,12
Test Corrections (Due Monday, 1/14)

Ch 17 Test - Wed, 1/9