IB Practice - Calculus - Differentiation (V2 Legacy)

1. If \(2x^2 - 3y^2 = 2\), find the two values of \(\frac{dy}{dx}\) when \(x = 5\).

\[
\text{Working:} \\
\text{Answer:} \\
………………………………………….. \\
\text{(Total 4 marks)}
\]

2. Differentiate \(y = \arccos (1 - 2x^2)\) with respect to \(x\), and simplify your answer.

\[
\text{Working:} \\
\text{Answer:} \\
………………………………………….. \\
\text{(Total 4 marks)}
\]

3. Give exact answers in this part of the question.
   The temperature \(g(t)\) at time \(t\) of a given point of a heated iron rod is given by
   \[g(t) = \frac{\ln t}{\sqrt{t}}, \quad \text{where} \ t > 0.\]
   
   (a) Find the interval where \(g'(t) > 0\).
   
   (b) Find the interval where \(g''(t) > 0\) and the interval where \(g''(t) < 0\).
   
   (c) Find the value of \(t\) where the graph of \(g(t)\) has a point of inflexion.
   
   (d) Let \(t^*\) be a value of \(t\) for which \(g'(t^*) = 0\) and \(g''(t^*) < 0\). Find \(t^*\).
   
   (e) Find the point where the normal to the graph of \(g(t)\) at the point \((t^*, g(t^*))\) meets the \(t\)-axis.

\[(\text{Total 18 marks)}\]

4. Let \(f(x) = \ln |x^5 - 3x^2|, -0.5 < x < 2, x \neq a, x \neq b;\) \((a, b\) are values of \(x\) for which \(f(x)\) is not defined).
   
   (a) (i) Sketch the graph of \(f(x)\), indicating on your sketch the number of zeros of \(f(x)\). Show also the position of any asymptotes.
   
   (ii) Find all the zeros of \(f(x)\), (that is, solve \(f(x) = 0\)).
   
   (b) Find the exact values of \(a\) and \(b\).
   
   (c) Find \(f(x)\), and indicate clearly where \(f'(x)\) is not defined.
   
   (d) Find the exact value of the \(x\)-coordinate of the local maximum of \(f(x)\), for \(0 < x < 1.5\). (You may assume that there is no point of inflexion.)
   
   (e) Write down the definite integral that represents the area of the region enclosed by \(f(x)\) and the \(x\)-axis. (Do not evaluate the integral.)

\[(\text{Total 16 marks)}\]

5. Differentiate from first principles \(f(x) = \cos x\).

\[(\text{Total 8 marks)}\]

6. For the function \(f : x \mapsto x^2 \ln x, x > 0,\) find the function \(f'\), the derivative of \(f\) with respect to \(x\).

\[(\text{Total 3 marks)}\]
7. For the function \( f: x \mapsto \frac{1}{2} \sin 2x + \cos x \), find the possible values of \( \sin x \) for which \( f'(x) = 0 \).

Working: .................................................................................................................. (Total 3 marks)

Answer: ..................................................................................................................

8. For what values of \( m \) is the line \( y = mx + 5 \) a tangent to the parabola \( y = 4 - x^2 \)?

Working: ..................................................................................................................

Answer: ..................................................................................................................

(Total 3 marks)

9. The tangent to the curve \( y^2 - x^3 \) at the point \( P(1, 1) \) meets the \( x \)-axis at \( Q \) and the \( y \)-axis at \( R \).

Find the ratio \( PQ : QR \).

Working: ..................................................................................................................

Answer: ..................................................................................................................

(Total 3 marks)

10. (a) Sketch and label the curves

\[ y = x^2 \text{ for } -2 \leq x \leq 2, \text{ and } y = -\frac{1}{2} \ln x \text{ for } 0 < x \leq 2. \]

(b) Find the \( x \)-coordinate of \( P \), the point of intersection of the two curves.

(c) If the tangents to the curves at \( P \) meet the \( y \)-axis at \( Q \) and the \( x \)-axis at \( R \), calculate the area of the triangle \( PQR \).

(d) Prove that the two tangents at the points where \( x = a, a > 0 \), on each curve are always perpendicular.

(Total 14 marks)

11. (a) Let \( y = \frac{a + b \sin x}{b + a \sin x} \), where \( 0 < a < b \).

(i) Show that \( \frac{dy}{dx} = \frac{(b^2 + a^2) \cos x}{(b + a \sin x)^2} \).

(ii) Find the maximum and minimum values of \( y \).

(iii) Show that the graph of \( y = \frac{a + b \sin x}{b + a \sin x} \), \( 0 < a < b \) cannot have a vertical asymptote.

(b) For the graph of \( y = \frac{4 + 5 \sin x}{5 + 4 \sin x} \) for \( 0 \leq x \leq 2\pi \),

(i) write down the \( y \)-intercept;

(ii) find the \( x \)-intercepts \( m \) and \( n \), (where \( m < n \)) correct to four significant figures;

(iii) sketch the graph.

(c) The area enclosed by the graph of \( y = \frac{4 + 5 \sin x}{5 + 4 \sin x} \) and the \( x \)-axis from \( x = 0 \) to \( x = n \) is denoted by \( A \). Write down, but do not evaluate, an expression for the area \( A \).

(Total 17 marks)
12. If \( f(x) = \ln(2x - 1), x > \frac{1}{2} \), find
   (a) \( f'(x) \);
   (b) the value of \( x \) where the gradient of \( f(x) \) is equal to \( x \).

   **Working:**
   **Answers:**
   (a) ............................................................
   (b) ............................................................

   **(Total 3 marks)**

13. Find the \( x \)-coordinate, between –2 and 0, of the point of inflexion on the graph of the function \( f : x \mapsto x^2 e^x \). Give your answer to 3 decimal places.

   **Working:**
   **Answer:** ............................................................

   **(Total 3 marks)**

14. Find the gradient of the tangent to the curve \( 3x^2 + 4y^2 = 7 \) at the point where \( x = 1 \) and \( y > 0 \).

   **Working:**
   **Answer:** ............................................................

   **(Total 3 marks)**

15. The function \( f \) is given by \( f : x \mapsto e^{(1 + \sin \pi x)}, x \geq 0 \).
   (a) Find \( f'(x) \).
   Let \( x_0 \) be the value of \( x \) where the \( (n + 1) \)th maximum or minimum point occurs, \( n \in \mathbb{N} \). (ie \( x_0 \) is the value of \( x \) where the first maximum or minimum occurs, \( x_1 \) is the value of \( x \) where the second maximum or minimum occurs, etc).
   (b) Find \( x_n \) in terms of \( n \).

   **Working:**
   **Answers:**
   (a) ............................................................
   (b) ............................................................

   **(Total 3 marks)**

16. Let \( f(x) = x \sqrt{(x^2 - 1)^2}, -1.4 \leq x \leq 1.4 \)
   (a) **Sketch** the graph of \( f(x) \). (An exact scale diagram is **not** required.)
   On your graph indicate the approximate position of
   (i) each zero;
   (ii) each maximum point;
   (iii) each minimum point.

   (b) (i) Find \( f'(x) \), clearly stating its domain.
   (ii) Find the \( x \)-coordinates of the maximum and minimum points of \( f(x) \), for \(-1 < x < 1\).

   (c) Find the \( x \)-coordinate of the point of inflexion of \( f(x) \), where \( x > 0 \), giving your answer correct to **four** decimal places.

   **Working:**
   **(Total 13 marks)**

17. The line \( y = 16x - 9 \) is a tangent to the curve \( y = 2x^3 + ax^2 + bx - 9 \) at the point (1,7). Find the values of \( a \) and \( b \).

   **(Total 3 marks)**
18. Consider the function \( y = \tan x - 8 \sin x \).
   (a) Find \( \frac{dy}{dx} \).
   (b) Find the value of \( \cos x \) for which \( \frac{dy}{dx} = 0 \).

**Working:**

**Answers:**

(a) ..........................................................

(b) ..........................................................

(Total 3 marks)

19. Consider the tangent to the curve \( y = x^3 + 4x^2 + x - 6 \).
   (a) Find the equation of this tangent at the point where \( x = -1 \).
   (b) Find the coordinates of the point where this tangent meets the curve again.

**Working:**

**Answers:**

(a) ..........................................................

(b) ..........................................................

(Total 3 marks)

20. Let \( y = \sin (kx) - kx \cos (kx) \), where \( k \) is a constant.
    Show that \( \frac{dy}{dx} = k^2 x \sin (kx) \).

(Total 3 marks)

21. Consider the function \( f(x) = \frac{1}{x^x} \), where \( x \in \mathbb{R}^+ \).
   (a) Show that the derivative \( f'(x) = f(x) \left( \frac{1-\ln x}{x^2} \right) \).

   (b) Sketch the function \( f(x) \), showing clearly the local maximum of the function and its horizontal asymptote. You may use the fact that \( \lim_{x \to \infty} \frac{\ln x}{x} = 0 \).

   (c) Find the Taylor expansion of \( f(x) \) about \( x = e \), up to the second degree term, and show that this polynomial has the same maximum value as \( f(x) \) itself.

(Total 13 marks)

22. A curve has equation \( xy^3 + 2x^2 y = 3 \). Find the equation of the tangent to this curve at the point \( (1, 1) \).

**Working:**

**Answer:**

..........................................................

(Total 6 marks)

23. The function \( f \) is defined by \( f(x) = \frac{x^2 - x + 1}{x^2 + x + 1} \).
   (a) (i) Find an expression for \( f'(x) \), simplifying your answer.
       (ii) The tangents to the curve of \( f(x) \) at points A and B are parallel to the \( x \)-axis. Find the coordinates of A and of B.

   (b) (i) Sketch the graph of \( y = f'(x) \).
       (ii) Find the \( x \)-coordinates of the three points of inflexion on the graph of \( f \).
(c) Find the range of
   (i) $f$;
   (ii) the composite function $f \circ f$.

24. Air is pumped into a spherical ball which expands at a rate of 8 cm$^3$ per second (8 cm$^3$ s$^{-1}$). Find the exact rate of increase of the radius of the ball when the radius is 2 cm.

   Working:
   
   Answer: .......................................................... 

   (Total 6 marks)

25. A curve has equation $x^3 y^2 = 8$. Find the equation of the normal to the curve at the point (2, 1).

   Working:
   
   Answer: .......................................................... 

   (Total 6 marks)

26. The function $f$ is defined by $f(x) = \frac{x^2}{2^x}$, for $x > 0$.

   (a) (i) Show that $f'(x) = \frac{2x - x^2 \ln 2}{2^x}$

   (ii) Obtain an expression for $f''(x)$, simplifying your answer as far as possible.

   (b) (i) Find the exact value of $x$ satisfying the equation $f'(x) = 0$

   (ii) Show that this value gives a maximum value for $f(x)$.

   (c) Find the $x$-coordinates of the two points of inflexion on the graph of $f$.

   (Total 12 marks)

27. Consider the function $f(t) = 3 \sec^2 t + 5t$.

   (a) Find $f'(t)$.

   (b) Find the exact values of

   (i) $f'(\pi)$;

   (ii) $f''(\pi)$;

   Working:
   
   Answers:
   (a) .......................................................... 

   (b) (i) ..........................................................

   (ii) ..........................................................

   (Total 6 marks)

28. Consider the equation $2xy^2 = x^2 y + 3$.

   (a) Find $y$ when $x = 1$ and $y < 0$.

   (b) Find $\frac{dy}{dx}$ when $x = 1$ and $y < 0$.

   Working:
   
   Answers:
   (a) .......................................................... 

   (b) ..........................................................

   (Total 6 marks)
29. Let \( y = e^{3x} \sin (\pi x) \).
   
   (a) Find \( \frac{dy}{dx} \).
   
   (b) Find the smallest positive value of \( x \) for which \( \frac{dy}{dx} = 0 \).

   **Working:**
   
   **Answers:**
   
   (a) ..................................................................
   
   (b) ..................................................................

   (Total 6 marks)

30. An airplane is flying at a constant speed at a constant altitude of 3 km in a straight line that will take it directly over an observer at ground level. At a given instant the observer notes that the angle \( \theta \) is \( \frac{1}{3} \pi \) radians and is increasing at \( \frac{1}{60} \) radians per second. Find the speed, in kilometres per hour, at which the airplane is moving towards the observer.

   **Working:**
   
   **Answer:**
   
   .........................................................................

   (Total 6 marks)

31. A curve has equation \( f(x) = \frac{a}{b + e^{-cx}} \), \( a \neq 0, b > 0, c > 0 \).
   
   (a) Show that \( f''(x) = \frac{ac^2e^{-cx}(e^{-cx} - b)}{(b + e^{-cx})^3} \).
   
   (b) Find the coordinates of the point on the curve where \( f''(x) = 0 \).
   
   (c) Show that this is a point of inflexion.

   (Total 8 marks)

32. The point \( P(1, p) \), where \( p > 0 \), lies on the curve \( 2x^2y + 3y^2 = 16 \).
   
   (a) Calculate the value of \( p \).
   
   (b) Calculate the gradient of the tangent to the curve at \( P \).

   **Working:**
   
   **Answers:**
   
   (a) ..................................................................
   
   (b) ..................................................................

   (Total 6 marks)

33. The function \( f \) is defined by \( f : x \mapsto 3^x \).
   
   Find the solution of the equation \( f''(x) = 2 \).

   (Total 6 marks)
34. The following diagram shows an isosceles triangle ABC with AB = 10 cm and AC = BC. The vertex C is moving in a direction perpendicular to (AB) with speed 2 cm per second.

![Triangle Diagram]

Calculate the rate of increase of the angle \( \angle CAB \) at the moment the triangle is equilateral.

**Working:**

**Answer:** ..........................................................................

(Total 6 marks)

35. If \( y = \ln (2x - 1) \), find \( \frac{d^2y}{dx^2} \).

**Working:**

**Answer:** ..........................................................................

(Total 6 marks)

36. Find the equation of the normal to the curve \( x^3 + y^3 - 9xy = 0 \) at the point \((2, 4)\).

**Working:**

**Answer:** ..........................................................................

(Total 6 marks)

37. The function \( f' \) is given by \( f'(x) = 2\sin \left( 5x - \frac{\pi}{2} \right) \).

(a) Write down \( f''(x) \).

(b) Given that \( f \left( \frac{\pi}{2} \right) = 1 \), find \( f(x) \).

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(Total 6 marks)

38. Find the gradient of the normal to the curve \( 3x^2y + 2xy^2 = 2 \) at the point \((1, -2)\).

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(Total 6 marks)

39. The function \( f \) is given by \( f(x) = \frac{x^3 + 2}{x} \), \( x \neq 0 \). There is a point of inflexion on the graph of \( f \) at the point P. Find the coordinates of P.

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(Total 6 marks)

40. An experiment is carried out in which the number \( n \) of bacteria in a liquid, is given by the formula \( n = 650 e^{kt} \), where \( t \) is the time in minutes after the beginning of the experiment and \( k \) is a constant. The number of bacteria doubles every 20 minutes. Find

(a) the exact value of \( k \);

(b) the rate at which the number of bacteria is increasing when \( t = 90 \).

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(Total 6 marks)
41. Let \( f(x) = \frac{x^2 + 5x + 5}{x + 2}, x \neq -2 \).
   
   (a) Find \( f''(x) \).
   
   (b) Solve \( f'(x) > 2 \).  

   (Total 6 marks)

42. The function \( f \) is defined by \( f(x) = e^{px}(x + 1) \), here \( p \in \mathbb{R} \).
   
   (a) (i) Show that \( f'(x) = e^{px}(p(x + 1) + 1) \).
   
   (ii) Let \( f^{(n)}(x) \) denote the result of differentiating \( f(x) \) with respect to \( x \), \( n \) times.
   
   Use mathematical induction to prove that 
   
   \[ f^{(n)}(x) = p^{n-1}e^{px}(p(x + 1) + n), n \in \mathbb{Z}^+ \].  

   (b) When \( p = \sqrt{3} \), there is a minimum point and a point of inflexion on the graph of \( f \). Find the exact value of the \( x \)-coordinate of
   
   (i) the minimum point;
   
   (ii) the point of inflexion.

   (4)

   (c) Let \( p = \frac{1}{2} \). Let \( R \) be the region enclosed by the curve, the \( x \)-axis and the lines \( x = -2 \) and \( x = 2 \).
   
   Find the area of \( R \).  

   (2)

   (Total 13 marks)

43. The diagram shows a trapezium OABC in which OA is parallel to CB. O is the centre of a circle radius \( r \) cm. A, B and C are on its circumference. Angle \( \hat{O\hat{C}B} = \theta \).

   Let \( T \) denote the area of the trapezium OABC.
   
   (a) Show that \( T = \frac{r^2}{2} \sin \theta + \sin 2\theta \).  

   (4)

   For a fixed value of \( r \), the value of \( T \) varies as the value of \( \theta \) varies.
   
   (b) Show that \( T \) takes its maximum value when \( \theta \) satisfies the equation 
   
   \( 4\cos^2\theta + \cos \theta - 2 = 0 \), and verify that this value of \( T \) is a maximum.  

   (5)

   (c) Given that the perimeter of the trapezium is 75 cm, find the maximum value of \( T \).  

   (6)

   (Total 15 marks)
44. Let \( f \) be a cubic polynomial function. Given that \( f(0) = 2, f'(0) = -3, f(1) = f'(1) \) and \( f''(-1) = 6 \), find \( f(x) \).

(Total 6 marks)

45. (a) Write down the term in \( x^r \) in the expansion of \((x + h)^n\), where \( 0 \leq r \leq n, n \in \mathbb{Z}^+ \).

(b) Hence differentiate \( x^n, n \in \mathbb{Z}^+ \), from first principles.

(c) Starting from the result \( x^n \times x^{-n} = 1 \), deduce the derivative of \( x^n, n \in \mathbb{Z}^+ \).

(Total 10 marks)

46. Let \( f(x) = \cos^3(4x + 1), 0 \leq x \leq 1 \).

(a) Find \( f'(x) \).

(b) Find the exact values of the three roots of \( f'(x) = 0 \).

(Total 6 marks)

47. Given that \( 3^x y = x^3 + 3y \), find \( \frac{dy}{dx} \).

(Total 6 marks)

48. Let \( f \) be the function defined for \( x > -\frac{1}{3} \) by \( f(x) = \ln(3x + 1) \).

(a) Find \( f'(x) \).

(b) Find the equation of the normal to the curve \( y = f(x) \) at the point where \( x = 2 \). Give your answer in the form \( y = ax + b \) where \( a, b \in \mathbb{R} \).

(Total 6 marks)

49. Let \( y = \cos \theta + i \sin \theta \).

(a) Show that \( \frac{dy}{d\theta} = iy \).

[You may assume that for the purposes of differentiation and integration, \( i \) may be treated in the same way as a real constant.]

(b) Hence show, using integration, that \( y = e^{i\theta} \).

(c) Use this result to deduce de Moivre’s theorem.

(d) (i) Given that \( \frac{\sin 6\theta}{\sin \theta} = a \cos^5 \theta + b \cos^3 \theta + c \cos \theta, \) where \( \sin \theta \neq 0 \), use de Moivre’s theorem with \( n = 6 \) to find the values of the constants \( a, b \) and \( c \).

(ii) Hence deduce the value of \( \lim_{\theta \to 0} \frac{\sin 6\theta}{\sin \theta} \).

(Total 20 marks)
50. Let \( f(x) = x \ln x - x, x > 0 \).
   (a) Find \( f'(x) \).
   (b) Using integration by parts find \( \int (\ln x)^2 \, dx \).

51. Let \( y = x \arcsin x, x \in [-1, 1] \). Show that \( \frac{d^2 y}{dx^2} = \frac{2-x^2}{(1-x^2)^{3/2}} \).

52. Given that \( e^{xy} - y^2 \ln x = e \) for \( x \geq 1 \), find \( \frac{dy}{dx} \) at the point \((1, 1)\).

53. The following table shows the values of two functions \( f \) and \( g \) and their first derivatives when \( x = 1 \) and \( x = 0 \).

<table>
<thead>
<tr>
<th></th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( g(x) )</th>
<th>( g'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>1</td>
<td>-4</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
<td>3</td>
<td>-1</td>
<td>2</td>
</tr>
</tbody>
</table>

(a) Find the derivative of \( \frac{3f(x)}{g(x)-1} \) when \( x = 0 \).
(b) Find the derivative of \( f(g(x) + 2x) \) when \( x = 1 \).

54. The function \( f \) is defined by \( f(x) = \frac{2x}{x^2 + 6} \) for \( x \geq b \) where \( b \in \mathbb{R} \).

(a) Show that \( f'(x) = \frac{12-2x^2}{(x^2 + 6)^2} \).
(b) Hence find the smallest exact value of \( b \) for which the inverse function \( f^{-1} \) exists. Justify your answer.

55. Consider the curve with equation \( x^2 + xy + y^2 = 3 \).
   (a) Find in terms of \( k \), the gradient of the curve at the point \((-1, k)\).
   (b) Given that the tangent to the curve is parallel to the \( x \)-axis at this point, find the value of \( k \).

(Total 6 marks)

(Total 6 marks)

(Total 6 marks)

(Total 6 marks)

(Total 6 marks)
56. André wants to get from point A located in the sea to point Y located on a straight stretch of beach. P is the point on the beach nearest to A such that AP = 2 km and PY = 2 km. He does this by swimming in a straight line to a point Q located on the beach and then running to Y.

When André swims he covers 1 km in \(5\sqrt{5}\) minutes. When he runs he covers 1 km in 5 minutes.

(a) If \(PQ = x\) km, \(0 \leq x \leq 2\), find an expression for the time \(T\) minutes taken by André to reach point Y.

(b) Show that \(\frac{dT}{dx} = \frac{5\sqrt{5}x}{\sqrt{x^2 + 4}} - 5\).

(c) (i) Solve \(\frac{dT}{dx} = 0\).

(ii) Use the value of \(x\) found in part (c) (i) to determine the time, \(T\) minutes, taken for André to reach point Y.

(iii) Show that \(\frac{d^2T}{dx^2} = \frac{20\sqrt{5}}{(x^2 + 4)^{3/2}}\) and hence show that the time found in part (c) (ii) is a minimum.

(Total 18 marks)

57. Find the gradient of the tangent to the curve \(x^3 y^2 = \cos (\pi y)\) at the point \((-1, 1)\).

(Total 12 marks)
IB Practice - Calculus - Differentiation: (V2 Legacy) MarkScheme

1. By implicit differentiation,
\[
\frac{d}{dx} (2x^2 - 3y^2 = 2) \Rightarrow 4x - 6y \frac{dy}{dx} = 0
\]
(M1)
\[
\therefore \frac{dy}{dx} = \frac{4x}{6y} = \frac{2x}{3y}
\]
(A1)

When \( x = 5 \), \( 50 - 3y^2 = 2 \)
\[
\therefore y^2 = 16
\]
\[
\therefore y = \pm 4
\]

Therefore \( \frac{dy}{dx} = \pm \frac{5}{6} \)

(C2)

Note: This can be done explicitly.

2. Given \( y = \arccos (1 - 2x^2) \)
then
\[
\frac{dy}{dx} = -\frac{1}{\sqrt{1 - (1 - 2x^2)^2}} \times -4x
\]
(M1)
\[
\frac{dy}{dx} = \frac{4x}{\sqrt{1 - 4x^2 + 4x^4}}
\]
(M1)
\[
\therefore \frac{dy}{dx} = \frac{4x}{\sqrt{4x^2 - 4x^4}}
\]
(A2)

OR
\[
\cos y = 1 - 2x^2
\]
\[
-\sin y \frac{dy}{dx} = -4x
\]
\[
\frac{dy}{dx} = \frac{4x}{\sqrt{1 - (1 - 2x^2)^2}}
\]
(M1)
\[
\frac{dy}{dx} = \frac{4x}{\sqrt{4x^2 - 4x^4}}
\]
(A2)

3. (a) \( g(t) = \frac{\ln t}{\sqrt{t}} \). So \( g'(t) = \frac{2 - \ln t}{2t^{3/2}} \).

Hence, \( g'(t) > 0 \) when \( 2 > \ln t \) or \( \ln t < 2 \) or \( t < e^2 \). (M1)

Since the domain of \( g(t) \) is \( \{ t: t > 0 \} \), \( g'(t) > 0 \) when \( 0 < t < e^2 \). (A1) 4

(b) Since \( g'(t) = \frac{2 - \ln t}{2t^{3/2}} \), \( g''(t) = \frac{-2\sqrt{t} - 3\sqrt{t}(2 - \ln t)}{4t^3} \)

\[
= -\frac{\sqrt{t}[8 - 3\ln t]}{4t^3}
\]
(A1)

Hence \( g''(t) > 0 \) when \( 8 - 3 \ln t < 0 \) or \( t > e^{8/3} \). (M1)

Similarly, \( g''(t) < 0 \) when \( 0 < t < e^{8/3} \). (A1) 5

(c) \( g''(t) = 0 \) when \( t = 0 \) or \( 8 = 3 \ln t \).

Since, the domain of \( g \) is \( \{ t: t > 0 \} \), \( g''(t) = 0 \) when \( t = e^{8/3} \). (M1)

Since \( g''(t) > 0 \) when \( t > e^{8/3} \) and \( g''(t) < 0 \) when \( t < e^{8/3} \),  
(M1)
\[ \left( e^{8/3}, \frac{8}{3} e^{-4/3} \right) \] is the point of inflexion. The required value of \( t \) is \( e^{8/3} \). (A1) 3

**Note:** Award (A1) for evaluating \( t \) as \( e^{8/3} \).

\( g'(t) = 0 \) when \( \ln t = 2 \) or \( t = e^2 \).

Also \( g''(e^2) = -\frac{\sqrt{e^2} [8 - 3 \ln e^2]}{4e^3} = -\frac{1}{2e^3} < 0 \) (M1)

Hence \( t^* = e^2 \) (A1) 3

At \( (t^*, g(t^*)) \) the tangent is horizontal. (M1)

So the normal at the point \( (t^*, g(t^*)) \) is the line \( t = t^* \). (M1)

Thus, it meets the \( t \) axis at the point \( t^* = e^2 \) and hence the point is \((e^2, 0)\). (A1) 3

4. (a) (i) \[ y = \ln |x^5 - 3x^2| \]

\[ \begin{array}{c}
\text{asymptote} \\
\text{asymptote}
\end{array} \]

**Note:** Award (G1) for correct shape, including three zeros, and (G1) for both asymptotes (G2)

(ii) \( f(x) = 0 \) for \( x = 0.599, 1.35, 1.51 \) (G1)(G1)(G
5

(b) \( f(x) \) is undefined for \( (x^5 - 3x^2) = 0 \) (M1)

\( x^2(x^3 - 3) = 0 \)

Therefore, \( x = 0 \) or \( x = 3^{1/3} \) (A2) 3

(c) \[ f'(x) = \frac{5x^4 - 6x}{x^5 - 3x^2} \left( \frac{5x^3 - 6}{x^4 - 3x} \right) \] (M1)(A1)

\( f'(x) \) is undefined at \( x = 0 \) and \( x = 3^{1/3} \) (A1) 3

(d) For the \( x \)-coordinate of the local maximum of \( f(x) \), where \( 0 < x < 1.5 \) put \( f'(x) = 0 \) (R1)

\[ 5x^3 - 6 = 0 \] (M1)

\[ x = \left( \frac{6}{5} \right)^{1/3} \] (A1) 3

(e) The required area is \[ A = \int_{0.599}^{1.35} f(x) \, dx \] (A2) 2

**Note:** Award (A1) for each correct limit.

5. Using first principles
\[ f'(x) = \lim_{h \to 0} \left( \frac{f(x + h) - f(x)}{h} \right) \]
\[ = \lim_{h \to 0} \left( \frac{\cos(x + h) - \cos(x)}{h} \right) \quad \text{(M1)} \]
\[ = \lim_{h \to 0} \left( \frac{\cos(x) \cos(h) - \sin(x) \sin(h) - \cos(x)}{h} \right) \quad \text{(M1)(A1)} \]
\[ = \cos(x) \lim_{h \to 0} \left( \frac{\cos(h) - 1}{h} \right) - \sin(x) \lim_{h \to 0} \left( \frac{\sin(h)}{h} \right) \quad \text{(M1)(A1)} \]

But \( \lim_{h \to 0} \left( \frac{\sin(h)}{h} \right) = 1 \) and \( \lim_{h \to 0} \left( \frac{\cos(h) - 1}{h} \right) = 0 \) \( \text{(C1)(C1)} \)

Therefore, \( f'(x) = -\sin x \) \( \text{(A1)} \)

OR

\[ f'(x) = \lim_{h \to 0} \left( \frac{f(x + h) - f(x)}{h} \right) \]
\[ = \lim_{h \to 0} \left( \frac{\cos(x + h) - \cos(x)}{h} \right) \quad \text{(M1)} \]
\[ = \lim_{h \to 0} \left( \frac{-2 \sin(x + \frac{1}{2}h) \sin(\frac{1}{2}h)}{2h} \right) \quad \text{(using any method)} \quad \text{(M1)(A2)} \]
\[ = \lim_{h \to 0} \left( -\sin \left( x + \frac{1}{2} \right) \sin \left( \frac{1}{2}h \right) \right) \quad \text{(M1)} \]

But \( \lim_{h \to 0} \left( \frac{\sin(h)}{h} \right) = 1 \) and \( \lim_{h \to 0} \left( -\sin \left( x + \frac{1}{2}h \right) \right) = -\sin x \) \( \text{(C2)} \)

Therefore, \( f'(x) = -\sin x \) \( \text{(A1)} \)

6. \( f(x) = x^2 \ln x \)
\( f'(x) = 2x \ln x + x^2 \left( \frac{1}{x} \right) \quad \text{(M1)(M1)} \)
\[ = 2x \ln x + x \quad \text{(A1)} \]
\[ f' : x \mapsto 2x \ln x + x \quad \text{(C3)} \]

7. \( f(x) = \frac{1}{2} \sin 2x + \cos x \)
\( f'(x) = \cos 2x - \sin x \quad \text{(M1)} \)
\[ = 1 - 2 \sin^2 x - \sin x \quad \text{(M1)} \]
\[ = (1 + \sin x)(1 - 2 \sin x) \quad \text{(M1)} \]
\[ = 0 \text{ when } \sin x = -1 \text{ or } \frac{1}{2} \quad \text{(A1)(C3)} \]

\[ 8 \]

\[ 3 \]
8. **Method 1:**

\[ y = 4 - x^2 \]

\[ \frac{dy}{dx} = -2x = m \text{ when } x = \frac{-m}{2} \]  

(M1)

Thus, \( \left( \frac{-m}{2}, 4 - \frac{m^2}{4} \right) \) lies on \( y = mx + 5 \).

(R1)

Then, \( 4 - \frac{m^2}{4} = -\frac{m^2}{2} + 5 \), so \( m^2 = 4 \)

\[ m = \pm 2. \]  

(A1)

(C3)

**Method 2:**

For intersection: \( mx + 5 = 4 - x^2 \)

(M1)

For tangency: discriminant = 0

(M1)

Thus, \( m^2 - 4 = 0 \)

\[ m = \pm 2. \]  

(A1)

(C3)

[3]

9. \( y^2 = x^3 \) so \( 2y \frac{dy}{dx} = 3x^2 \).

At \( P(1, 1) \), \( \frac{dy}{dx} = \frac{3}{2} \).

(M1)

The tangent is \( 3x - 2y = 1 \), giving \( Q = \left( \frac{1}{3}, 0 \right) \) and \( R = \left( 0, -\frac{1}{2} \right) \).

(A1)

Therefore, \( PQ : QR = \frac{2}{3} : \frac{1}{3} \) or \( 1 : \frac{1}{2} \)

\[ = 2 : 1. \]  

(A1)

(C3)

[3]

10. (a)

\[ \frac{dy}{dx} = 2x \]

(C2) 2

Note: Award (C1) for \( y = x^2 \), (C1) for \( y = -\frac{1}{2} \ln x \).

(b) \( x^2 + \frac{1}{2} \ln x = 0 \) when \( x = 0.548217 \).

Therefore, the \( x \)-coordinate of \( P \) is 0.548….

(G2) 2

(c) The tangent at \( P \) to \( y = x^2 \) has equation \( y = 1.0964x - 0.30054 \),

(G2)

and the tangent at \( P \) to \( y = -\frac{1}{2} \ln x \) has equation \( y = -0.91205x + 0.80054 \).

Thus, the area of triangle \( PQR = \frac{1}{2} (0.30052 + 0.80054)(0.5482) \)

\[ = 0.302 \text{ (3 sf)} \]  

(A1)

OR

\[ y = x^2 \Rightarrow \frac{dy}{dx} = 2x \]

(M1)
Therefore, the tangent at \((p, p^2)\) has equation \(2px - y = p^2\).  
\[
y = -\frac{1}{2} \ln x \Rightarrow \frac{dy}{dx} = -\frac{1}{2x}
\]
Therefore, the tangent at \((p, p^2)\) has equation \(x + 2py = p + 2p^3\).  
Thus, \(Q = (0, -p^2)\) and \(R = (0, p^2 + \frac{1}{2})\).  
Thus, the area of the triangle \(PQR\) 
\[
= \frac{1}{2} (2p^2 + \frac{1}{2})p
\]
\[
= 0.302 \text{ (3 sf)}
\]
\[(A1)\]

\[6\]

(d) \(y = x^2\) \(\Rightarrow\) when \(x = a\), \(\frac{dy}{dx} = 2a\)  
\[
y = -\frac{1}{2} \ln x \Rightarrow \text{when } x = a, \left(\frac{dy}{dx} = -\frac{1}{2a}\right) \quad \text{\((a > 0)\)}
\]

Now, \(2a) \left(-\frac{1}{2a}\right) = -1\) for all \(a > 0\).  
Therefore, the tangents to the curve at \(x = a\) on each curve are always perpendicular.  
\[(R1)(AG)\]

11. (a) (i) \(y = \frac{a + b \sin x}{b + a \sin x}, 0 < a < b\)  
\[
\frac{dy}{dx} = \frac{(b + a \sin x)(b \cos x) - (a + b \sin x) (a \cos x)}{(b + a \sin x)^2}
\]
\[
= \frac{b^2 \cos x + ab \sin x \cos x - a^2 \cos x - ab \sin x \cos x}{(b + a \sin x)^2}
\]
\[
= \frac{(b^2 - a^2) \cos x}{(b + a \sin x)^2}
\]
\[(M1)(C1)\]

(ii) \(\frac{dy}{dx} = 0 \Rightarrow \cos x = 0\) since \(b^2 - a^2 \neq 0\).  
This gives \(x = \frac{\pi}{2} (+\pi k, k \in \mathbb{Z})\)  
\[(M1)(C1)\]

When \(x = \frac{\pi}{2}\), \(y = \frac{a + b}{b + a} = 1\),  
and when \(x = \frac{3\pi}{2}\), \(y = \frac{a - b}{b - a} = -1\).  
Therefore, maximum \(y = 1\) and minimum \(y = -1\).  
\[(A2)\]

(iii) A vertical asymptote at the point \(x\) exists if and only if \(b + a \sin x = 0\).  
Then, since \(0 < a < b\), \(\sin x = -\frac{b}{a} < -1\), which is impossible.  
Therefore, no vertical asymptote exists.  
\[(AG)\]

(b) (i) \(y\)-intercept = 0.8  
\[(A1)\]

(ii) For \(x\)-intercepts, \(\sin x = -\frac{4}{5} \Rightarrow x = 4.069, 5.356\).  
\[(A2)\]

(iii)
12. (a) If \( f(x) = \ln(2x - 1) \),
Then \( f'(x) = \frac{2}{2x - 1} \)  
(A2)
(b) Put \( \frac{2}{2x - 1} = x \)
\[ \Rightarrow x = 1.28 \] (using a graphic display calculator or the quadratic formula)  
(A1)

13. If \( f(x) \Rightarrow x^2e^x \)
then \( f'(x) = x^2e^x + 2xe^x \)
\[ f''(x) = x^2e^x + 4xe^x + 2e^x = e^x(x^2 + 4x + 2) \]  
(A1)

For a point of inflexion solve \( f''(x) = 0 \)
\[ f''(x) = 0 \text{ at } x = -0.586 \] (using a graphic display calculator or the quadratic formula)  
(A1)

(Since \( f'(x) \neq 0 \) at this value, then it is a point of inflexion.)

Note: Some candidates may find the value of \( x \) from \( f'(x) \) by finding the minimum turning point using a graphic display calculator  

[3]

14. \( 3x^2 + 4y^2 = 7 \)
When \( x = 1, y = 1 \) (since \( y > 0 \))  
(M1)
\[ \frac{d}{dx} (3x^2 + 4y^2 = 7) \Rightarrow 6x + 8y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{3x}{4y} \]  
(A1)

The gradient where \( x = 1 \) and \( y = 1 \) is \(-\frac{3}{4}\)  
(A1)(C3)

OR
\[ 3x^2 + 4y^2 = 7 \]
\[ \Rightarrow y = \sqrt{\frac{7 - 3x^2}{4}}, \text{ since } y > 0 \]  
(M1)
\[ \frac{dy}{dx} = -\frac{3x}{2(7 - 3x^2)^{\frac{1}{2}}} \]  
(A1)
15. (a) \( f'(x) = \pi \cos (\pi x)e^{(1+\sin \pi x)} \)  
\[ \text{ (A1) (C1)} \]

(b) For maximum or minimum points, \( f'(x) = 0 \)
\[ \cos \pi x = 0 \]  
\[ \pi x = \frac{2k + 1}{2} \pi \]
then \( x_n = \frac{2n + 1}{2} \)  
\[ \text{ (A1) (C2) } \]

16. (a) \( f(x) = x \left(\frac{3}{2} \sqrt{(x^2 - 1)^2}\right) \)  
\[ \text{ (A4) } 4 \]

Notes: Award (A1) for the shape, including the two cusps (sharp points) at \( x = \pm 1 \).
(i) Award (A1) for the zeros at \( x = \pm 1 \) and \( x = 0 \).
(ii) Award (A1) for the maximum at \( x = -1 \) and the minimum at \( x = 1 \).
(iii) Award (A1) for the maximum at approx. \( x = 0.65 \), and the minimum at approx. \( x = -0.65 \)
There are no asymptotes.
The candidates are not required to draw a scale.

(b) (i) Let \( f(x) = x \left(\frac{3}{2} \sqrt{(x^2 - 1)^2}\right) \)
Then \( f'(x) = \frac{4}{3} x^2 (x^2 - 1)^{-\frac{1}{2}} + (x^2 - 1)^{\frac{1}{2}} \)  
\[ f'(x) = (x^2 - 1)^{-\frac{1}{2}} \left[ \frac{4}{3} x^2 + (x^2 - 1) \right] \]  
\[ f'(x) = (x^2 - 1)^{-\frac{1}{2}} \left( \frac{7}{3} x^2 - 1 \right) \] (or equivalent)  
\[ f'(x) = \frac{7x^2 - 3}{3(x^2 - 1)^{\frac{1}{2}}} \] (or equivalent)  
The domain is \(-1.4 \leq x \leq 1.4, x \neq \pm 1 \) (accept \(-1.4 < x < 1.4, x \neq \pm 1 \) \[ \text{ (A1) } \]

(ii) For the maximum or minimum points let \( f'(x) = 0 \)
\[ i.e. (7x^2 - 3) = 0 \] or use the graph.  
\[ \text{ (M1) } \]
Therefore, the \(x\)-coordinate of the maximum point is
\[
x = \frac{\sqrt{3}}{7} \quad \text{(or 0.655)} \quad \text{(A1)}
\]
and the \(x\)-coordinate of the minimum point is
\[
x = -\frac{\sqrt{3}}{7} \quad \text{(or -0.655)}. \quad \text{(A1)}
\]

**Notes:** Candidates may do this using a GDC, in that case award (M1)(G2).

(c) The \(x\)-coordinate of the point of inflexion is \(x = \pm 1.1339\) (G2)

**OR**

\[
f''(x) = \frac{4x(7x^2 - 9)}{9(x^2 - 1)^4}, \quad x \neq \pm 1 \quad \text{(M1)}
\]

For the points of inflexion let \(f''(x) = 0\) and use the graph,
\[
\text{ie } x = \frac{\sqrt{9}}{7} = 1.1339. \quad \text{(A1)} \quad \text{(2)}
\]

**Note:** Candidates may do this by plotting \(f'(x)\) and finding the \(x\)-coordinate of the minimum point. There are other possible methods.

17. For the curve, \(y = 7\) when \(x = 1\) \(\Rightarrow a + b = 14\), and

\[
\frac{\text{d}y}{\text{d}x} = 6x^2 + 2ax + b = 16 \quad \text{when } x = 1 \Rightarrow 2a + b = 10.
\]

Solving gives \(a = -4\) and \(b = 18\). (A1)(C3) [3]

18. (a) \[
\frac{\text{d}y}{\text{d}x} = \sec^2 x - 8 \cos x \quad \text{(A1)} \quad \text{(C1)}
\]

(b) \[
\frac{\text{d}y}{\text{d}x} = \frac{1 - 8 \cos^3 x}{\cos^2 x} \quad \text{(M1)}
\]

\[
\frac{\text{d}y}{\text{d}x} = 0
\]

\(\Rightarrow \cos x = \frac{1}{2} \quad \text{(A1)} \quad \text{(C2)} \quad \text{[3]}

19. **METHOD 1**

(a) The equation of the tangent is \(y = -4x - 8\). (G2)(C2)

(b) The point where the tangent meets the curve again is \((-2, 0)\). (G1)(C1)

**METHOD 2**

(a) \[
y = -4 \quad \text{and } \frac{\text{d}y}{\text{d}x} = 3x^2 + 8x + 1 = -4 \quad \text{at } x = -1.
\]

Therefore, the tangent equation is \(y = -4x - 8\). (A1)(C2)

(b) This tangent meets the curve when \(-4x - 8 = x^3 + 4x^2 + x - 6\) which gives \(x^3 + 4x^2 + 5x + 2 = 0 \Rightarrow (x + 1)^2(x + 2) = 0\).

The required point of intersection is \((-2, 0)\). (A1)(C1) [3]

20. \[
y = \sin (kx) - kx \cos (kx)
\]

\[
\frac{\text{d}y}{\text{d}x} = k \cos (kx) - k \{\cos (kx) + x[-k \sin (kx)]\} \quad \text{(M1)(C1)}
\]

\[
= k \cos (kx) - k \cos (kx) + k^2 x \sin (kx) \quad \text{(C1)}
\]

\[
= k^2 x \sin (kx) \quad \text{(AG)}
\]
21. (a) The derivative can be found by logarithmic differentiation. Let \( y = f(x) \).

\[
y = x^{\frac{1}{x}} \Rightarrow \ln y = \frac{1}{x} \ln x
\]

(M1)

\[
\frac{y'}{y} = -\frac{1}{x^2} \ln x + \frac{1}{x} \times \frac{1}{x} = \frac{1 - \ln x}{x^2}
\]

(M1)(M1)

\[
y' = \frac{1 - \ln x}{x^2}
\]

that is, \( f'(x) = f(x) \left( \frac{1 - \ln x}{x^2} \right) \) (AG) 3

(b) This function is defined for positive and real numbers only.

To find the exact value of the local maximum:

\[
y' = 0 \Rightarrow \ln x = 1 \Rightarrow x = e
\]

(M1)

\[
y = e^x
\]

(A1)

To find the horizontal asymptote:

\[
\lim_{x \to \infty} y = \lim_{x \to \infty} \ln x = 0
\]

(M1)(A1)

\[
\Rightarrow \lim_{x \to \infty} y = 1
\]

(c) By Taylor’s theorem we have

\[
P_2(x) = f(e) + f'(e)(x-e) + \frac{f''(e)}{2} (x-e)^2
\]

(A1)

\[
f''(x) = f'(x) \left( \frac{1 - \ln x}{x^2} \right) + f(x) \left( \frac{2 \ln x - 3}{x^3} \right)
\]

(M1)

Also, \( f'(e) = 0 \), and \( f''(e) = 0 + f(e) \left( \frac{2 - 3}{e^3} \right) = e^e \left( \frac{1}{e^3} \right) = \frac{1}{e^3} \) (M1)(A1)

hence \( P_2(x) = e^x - \frac{e^x}{2} (x-e)^2 \) which is a parabola with vertex

at \( x = e \) and \( P_2(e) = e^e = f(e) \) (R1)(AG) 5

22. \[
y^3 + 3xy^2 \frac{dy}{dx} + 4xy + 2x^2 \frac{dy}{dx} = 0
\]

(M1)(A1)
\[ \Rightarrow \frac{dy}{dx} = -\frac{\left(y^3 + 4xy\right)}{3xy^2 + 2x^2} \]  

(A1)

At (1,1), \( \frac{dy}{dx} = -1 \)  

(A1)

Equation of tangent is \( y - 1 = -1(x - 1) \) or \( x + y = 2 \)  

(A2)  

(C6)

23.  

(a)  

(i) \( f'(x) = \frac{(2x-1)(x^2 + x + 1) - (2x+1)(x^2 - x + 1)}{(x^2 + x + 1)^2} \)  

(M1)(A1)

\[ = \frac{2(x^2 - 1)}{(x^2 + x + 1)^2} \]  

(A1)

(ii) \( f'(x) = 0 \Rightarrow x = \pm 1 \)

A \( \left(1, \frac{1}{3}\right) \) B(-1, 3)  

or A \( (-1, 3) \) B \( \left(1, \frac{1}{3}\right) \)  

(A1)(A1)  

5

(b)  

(i)  

The graph of \( y = f(x) \) helps:

(G2)

Note: Award (G1) for general shape  
and (G1) for indication of scale.

(ii) The points of inflexion can be found by locating the max/min on the graph of \( f' \).  
This gives \( x = -1.53, -0.347, 1.88 \).  

OR

\[ f''(x) = -\frac{4(x^3 - 3x - 1)}{(x^2 + x + 1)^3} \]  

(M1)

\[ f''(x) = 0 \Rightarrow x^3 - 3x - 1 = 0 \]  

(A1)

\[ \Rightarrow x = 1.53, -0.347, 1.88 \]  

(G1)  

5

(c) The graph of \( y = f(x) \) helps:
(i) Range of \( f \) is \([\frac{1}{3}, 3]\). \( \text{(A1)(A1)} \)

(ii) We require the image set of \([\frac{1}{3}, 3]\).

\[
\begin{align*}
  f\left(\frac{1}{3}\right) &= \frac{1}{9} - \frac{1}{3} + 1 = \frac{7}{13},
  f(3) &= \frac{9 - 3 + 1}{9 + 3 + 1} = \frac{7}{13} \\
\end{align*}
\]

Range of \( g \) is \([\frac{1}{3}, \frac{7}{13}]\). \( \text{(A1)(A1) 5} \)

Note: Since the question did not specify exact ranges accept open intervals or numerical approximations (eg \(0.333, 0.538\)).

[15]

24. \[
\frac{dV}{dt} = 8 \, \text{(cm}^3\text{s}^{-1}), \quad V = \frac{4}{3} \pi r^3
\]

\[
\Rightarrow \frac{dV}{dr} = 4\pi r^2
\]

\[
\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = \frac{dV}{dr} \div \left( \frac{dV}{dr} \right)
\]

When \( r = 2 \), \[
\frac{dr}{dt} = 8 \div (4\pi \times 2^2)
\]

\[
= \frac{1}{2\pi} \, \text{(cm s}^{-1}) \quad \text{(do not accept 0.159)} \quad \text{(A1) (C6)}
\]

[6]

25. **METHOD 1**

\[
3x^2y^2 + x^3y \frac{dy}{dx} = 0
\]

(M1)(A1)

At \((2, 1)\), \[
12 + 16 \frac{dy}{dx} = 0
\]

\[
\Rightarrow \frac{dy}{dx} = -\frac{3}{4}
\]

(A1)

Gradient of normal = \(\frac{4}{3}\) \(\text{(A1)}\)

Equation of normal is \( y - 1 = \frac{4}{3} (x - 2) \) \(\text{(A1) (C6)}\)

**METHOD 2**

\[
y = 2\sqrt[3]{2x^{\frac{3}{2}}}
\]

\[
\frac{dy}{dx} = -3\sqrt[3]{2x^{\frac{5}{2}}}
\]

\[
= -\frac{3}{4} \quad \text{when } x = 2 \quad \text{(A1)}
\]
Gradient of normal = \frac{4}{3} \hspace{5cm} \text{(A1)}

Equation of normal is y – 1 = \frac{4}{3}(x – 2) \hspace{5cm} \text{(A1) (C6)}

26. (a) (i) \( f'(x) = \frac{2x \cdot 2^x - x^2 2^x \ln 2}{2^x} \)

\[ = \frac{2x - x^2 \ln 2}{2^x} \hspace{5cm} \text{(M1)(A1)} \]

\[ = \frac{2x - x^2 \ln 2}{2^x} \hspace{5cm} \text{(AG)} \]

(ii) \( f''(x) = \frac{2^x [2 - 2x \ln 2] - 2^x \ln 2 [2x - x^2 \ln 2]}{2^x} \)

\[ = \frac{x^2 (\ln 2)^2 - 4x \ln 2 + 2}{2^x} \hspace{5cm} \text{(M1)(A1)} \]

\[ = \frac{x^2 (\ln 2)^2 - 4x \ln 2 + 2}{2^x} \hspace{5cm} \text{(A1) 5} \]

\text{Note: Award the second (A1) for some form of simplification, eg accept} \frac{x \ln 2(x \ln 2 - 4) + 2}{2^x}. \hspace{5cm} \text{eg accept} \frac{x \ln 2(x \ln 2 - 4) + 2}{2^x}.

(b) (i) \( 2x - x^2 \ln 2 = 0 \) giving \( x = \frac{2}{\ln 2} \) \hspace{5cm} \text{(M1)(A1)}

\text{Note: Award (M1)(A0) for} x = 2.89. \hspace{5cm} \text{(M1)(A1)}

(ii) With this value of x, \( f''(x) = \frac{4 - 8 + 2}{x} \hspace{5cm} < 0 \)

\text{Therefore, a maximum.} \hspace{5cm} \text{(AG) 4}

(c) Points of inflexion satisfy \( f''(0) = 0 \), ie \( x^2 (\ln 2)^2 + 4x \ln 2 + 2 = 0 \)

\[ \Rightarrow x = \frac{4 \ln 2 \pm \sqrt{4(\ln 2)^2}}{2(\ln 2)^2} \]

\[ = \frac{2 \pm \sqrt{2}}{\ln 2} \hspace{5cm} (0.845, 4.93) \]

\text{OR} \hspace{5cm} \text{(M1)(G1)(G)}

\[ x = 0.845, 4.93 \]

\[ \text{1) } \hspace{5cm} \text{3} \]

27. (a) \text{METHOD 1}

\( f(t) = 3 \sec^2 t + 5t \)

\( f(t) = 3(\cos t)^{-2} + 5t \)

\( f(t) = -6(\cos t)^{-3}(\sin t) + 5 \)

\[ = \frac{6 \sin t}{\cos^3 t} + 5 \hspace{5cm} \text{(C2)} \]

\text{METHOD 2}

\( f''(t) = 3 \times 2 \sec t(\sec t \tan t) + 5 \)

\[ = 6 \sec^2 t \tan t + 5 = (6 \tan^2 t + 6 \tan t + 5) \hspace{5cm} \text{(C2)} \]
(b) \( f(\pi) = \frac{3}{\cos^2 \pi} + 5\pi \)  
\[ = 3 + 5\pi \]  
(M1)  
\[ = 3 + 5\pi \]  
(A1)  
(C2)  

\( f'(\pi) = \frac{6\sin \pi}{(\cos \pi)^3} + 5 \)  
\[ = 5 \]  
(M1)  
\[ = 5 \]  
(A1)  
(C2)  

[6]

28. \( 2xy^2 = x^2y + 3 \)  
(a) \( x = 1 \Rightarrow 2y^2 - y - 3 = 0 \)  
\[ y = \frac{3}{2} \text{ or } y = -1 \]  
\( y < 0 \Rightarrow y = -1 \)  
(M1)  
\[ y = \frac{3}{2} \text{ or } y = -1 \]  
(A1)  
(C2)  

(b) \( 2y^2 + 4xy \frac{dy}{dx} = 2xy + x^2 \frac{dy}{dx} \)  
\[ 1 \)  
\[ \left( \frac{dy}{dx} = \frac{2xy - 2y^2}{4xy - x^2} \right) \]  
\[ (1, -1) \Rightarrow \frac{dy}{dx} = \frac{4}{5} \]  
(M1)(M1)(A1)  
\[ (1, -1) \Rightarrow \frac{dy}{dx} = \frac{4}{5} \]  
(A1)  
(C4)  

[6]

29. \( y = e^{3x} \sin (\pi x) \)  
(a) \( \frac{dy}{dx} = 3e^{3x} \sin (\pi x) + \pi e^{3x} \cos (\pi x) \)  
\[ 1 \)  
\[ \left( \frac{dy}{dx} = 3e^{3x} \sin (\pi x) + \pi e^{3x} \cos (\pi x) \right) \]  
\[ (b) \]  
\[ 0 = e^{3x}(3 \sin (\pi x) + \pi \cos (\pi x)) \]  
\[ \tan (\pi x) = -\frac{\pi}{3} \]  
\[ \pi x = -0.80845 + \pi \]  
\[ x = 0.7426... \text{ (0.743 to 3 sf)} \]  
(M1)  
\[ \tan (\pi x) = -\frac{\pi}{3} \]  
(M1)  
\[ \pi x = -0.80845 + \pi \]  
(A1)  
(C3)  

[6]

30. \( \tan \theta = \frac{3}{x} \)  
\( \sec^2 \theta \frac{d\theta}{dt} = -3 \frac{dx}{x^2 \frac{dx}{dt}} \)  
(M1)  
\[ \sec^2 \theta \frac{d\theta}{dt} = -3 \frac{dx}{x^2 \frac{dx}{dt}} \]  
(M1)  
when \( \theta = \frac{\pi}{3}, x^2 = 3 \) and \( \sec^2 \theta = 4 \)  
\[ \frac{dx}{dt} = -\frac{x^2 \sec^2 \theta}{3} \frac{d\theta}{dt} \]  
\[ \frac{dx}{dt} = -3(4) \left( \frac{1}{60} \right) \]  
\[ \frac{dx}{dt} = -\frac{x^2 \sec^2 \theta}{3} \frac{d\theta}{dt} \]  
\[ \frac{dx}{dt} = -3(4) \left( \frac{1}{60} \right) \]  
\[ \frac{dx}{dt} = -3(4) \left( \frac{1}{60} \right) \]  
(A1)(A1)  

(M1)
\[ \frac{dx}{dt} = -\frac{1}{15} \text{ km s}^{-1} \]
\[ \frac{dx}{dr} = -240 \text{ km h}^{-1} \]

The airplane is moving towards him at 240 km h^{-1} \hspace{1cm} \text{(A1)}
\hspace{1cm} \text{(A1)} \hspace{1cm} \text{(C6)}

\text{Note: Award (C5) if the answer is given as } -240 \text{ km h}^{-1}. \hspace{1cm} [6]

31. \( f(x) = \frac{a}{b + e^{-cx}}, \ a \neq 0, \ b > 0, \ c > 0 \)
\hspace{1cm} \text{(a)} \ \ f'(x) = \frac{(b + e^{-cx})(0) - (a)(-ce^{-cx})}{(b + e^{-cx})^2} \hspace{1cm} \text{(M1)}
\hspace{1cm} = \frac{ace^{-cx}}{(b + e^{-cx})^2} \hspace{1cm} \text{(A1)}

\[ f''(x) = \frac{(b + e^{-cx})^2 (-ace^{-cx}) - (ace^{-cx})2(b + e^{-cx})(-ce^{-cx})}{(b + e^{-cx})^4} \]
\hspace{1cm} = \frac{-bace^{-cx} - ace^{-cx} + 2ace^{-cx}}{(b + e^{-cx})^3} \hspace{1cm} \text{(M1)}
\hspace{1cm} = \frac{ace^{-cx}(e^{-cx} - b)}{(b + e^{-cx})^3} \hspace{1cm} \text{(A1)} \hspace{1cm} \text{(AG) 4}

\hspace{1cm} \text{(b)} \ \ f''(x) = 0 \Rightarrow e^{-cx} = b
\hspace{1cm} \Rightarrow -cx = \ln b
\hspace{1cm} \Rightarrow x = -\frac{1}{c} \ln b \Rightarrow y = \frac{a}{b + e^{\ln b}} = \frac{a}{2b} \hspace{1cm} 2 \ a \ b =
\hspace{1cm} \text{So coordinate} = \left(-\frac{1}{c} \ln b, \frac{a}{2b}\right) \hspace{1cm} \text{(A1)(A1) 2}

\hspace{1cm} \text{(c)} \ Now e^{-cx} > b \ on \ one \ side \ of \ x = -\frac{1}{c} \ ln \ b \ and \ e^{-cx} < b \ on \ the \ other \ side. \hspace{1cm} \text{(R1)}
\hspace{1cm} \Rightarrow f''(x) \ changes \ sign \ at \ this \ point. \hspace{1cm} \text{(R1)}
\hspace{1cm} \Rightarrow \text{It \ is \ a \ point \ of \ inflexion.} \hspace{1cm} \text{(AG) 2} \hspace{1cm} \text{[8]}

32. \hspace{1cm} \text{(a)} \ \ 2p + 3p^2 = 16 \Rightarrow p = 2 \hspace{1cm} \text{(M1)(A1) (C2)}
\hspace{1cm} \text{Note: Do not penalize if } p = \frac{8}{3} \text{ also appears.}

\hspace{1cm} \text{(b)} \ \ 4xy + 2x^2 \frac{dy}{dx} + 6y \frac{dy}{dx} = 0 \hspace{1cm} \text{(A1)(A1)}
\hspace{1cm} \text{Note: Award (A1) for } 4xy + 2x^2 \frac{dy}{dx} \text{ and (A1) for } 6y \frac{dy}{dx} = 0.
\hspace{1cm} \text{at } P(1, 2), \ 8 + 2 \frac{dy}{dx} + 12 \frac{dy}{dx} = 0 \Rightarrow 14 \frac{dy}{dx} = -8 \hspace{1cm} \text{(A1)}
\hspace{1cm} \Rightarrow \text{gradient} = -\frac{4}{7} (= -0.571) \hspace{1cm} \text{(A1)(C4) [6]}
33. \( f(x) = 3^x \Rightarrow f'(x) = 3^x \ln 3 \)  
\( \Rightarrow f''(x) = 3^x (\ln 3)^2 \)  
\( 3^x (\ln 3)^2 = 2 \)  
\( 3^x = \frac{2}{(\ln 3)^2} \)  
\( x \ln 3 = \ln \left( \frac{2}{(\ln 3)^2} \right) \)  
\( x = \frac{\ln 2}{\ln 3} \)  
\( = 0.460 \)  
(M1)(A1)(C6)

34. Let \( h = \) height of triangle and \( \theta = \angle C \hat{A}B \).  
Then, \( h = 5 \tan \theta \)  
\( \frac{dh}{dt} = 5 \sec^2 \theta \times \frac{d\theta}{dt} \)  
Put \( \theta = \frac{\pi}{3} \).  
\( 2 = 5 \times 4 \times \frac{d\theta}{dt} \)  
\( \frac{d\theta}{dt} = \frac{1}{10} \) rad per sec \( \left( \text{Accept } \frac{180}{\pi} \text{ per second or } 5.73 \text{ per second} \right) \)  
(A1)(A1)(C6)

**Note:** Award (A1) for the correct value, and (A1) for the correct units.

35. \( y = \ln (2x - 1) \)  
\( \Rightarrow \frac{dy}{dx} = \frac{2}{2x - 1} \)  
(M1)(A1)  
\( \Rightarrow \frac{dy}{dx} = 2(2x - 1)^{-1} \)  
(A1)  
\( \Rightarrow \frac{d^2y}{dx^2} = -2(2x - 1)^{-2} \)  
(M1)(A1)  
\( \Rightarrow \frac{d^2y}{dx^2} = \frac{-4}{(2x - 1)^2} \) or \( -4(2x - 1)^{-2} \)  
(A1)(C6)

36. \( x^3 + y^3 - 9xy = 0 \)  
Differentiating w.r.t. \( x \)  
\( \Rightarrow 3x^2 + 3y^2 \frac{dy}{dx} - 9y - 9x \frac{dy}{dx} = 0 \)  
(A1)(A1)  

**Note:** Award (A1) for \( 3x^2 + 3y^2 \frac{dy}{dx} \), and (A1) for \(-9y - 9x \frac{dy}{dx} \).  

\( \Rightarrow \frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} \)  
(A1)

**EITHER**
at point (2, 4) gradient = 0.8.  
⇒ Gradient of normal = \(-1.25\)  
**OR**

Gradient of normal = \(-\frac{3y^2 + 9x}{9y - 3x^2}\)  
\(\text{at point (2, 4), gradient is}\ -1.25\)  

**THEN**

Equation of normal is given by
\[y - 4 = -1.25(x - 2)\] or
\[y = -1.25x + 6.5\] (**A1**)(**C6**)

37. (a) Using the chain rule \(f'(x) = \left(2 \cos\left(5x - \frac{\pi}{2}\right)\right)5\)  
\(= 10 \cos\left(5x - \frac{\pi}{2}\right)\)  
**M1**  
**A1** 2

(b) \(f(x) = \int f'(x) \, dx\)
\[= -\frac{2}{5} \cos\left(5x - \frac{\pi}{2}\right) + c\]  
**A1**

Substituting to find \(c\), \(f\left(\frac{\pi}{2}\right) = -\frac{2}{5} \cos\left(5 \left(\frac{\pi}{2}\right) - \frac{\pi}{2}\right) + c = 1\)  
**M1**

\[c = 1 + \frac{2}{5} \cos 2\pi = 1 + \frac{2}{5} = \frac{7}{5}\]  
\[f(x) = -\frac{2}{5} \cos\left(5x - \frac{\pi}{2}\right) + \frac{7}{5}\]  
**A1** **N2** 4

38. Attempting to differentiate implicitly  
\[3x^2y + 2xy^2 = 2 \Rightarrow 6xy + 3x^2 \frac{dy}{dx} + 2y^2 + 4xy \frac{dy}{dx} = 0\]  
**A1**

Substituting \(x = 1\) and \(y = -2\)  
\[-12 + 3 \frac{dy}{dx} + 8 - 8 \frac{dy}{dx} = 0\]  
\[\Rightarrow -5 \frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = -\frac{4}{5}\]  
**A1**

Gradient of normal is \(-\frac{5}{4}\).  
**A1** **N3**

39. **METHOD 1**

\[f'(x) = 4x^3 - \frac{2}{x^2}\]  
**M1**(**A1**)

\[f''(x) = 12x^2 + \frac{4}{x^3}\]  
**A1**

\[f'''(x) = 0\]  
**M1**

\[\Rightarrow x = -\frac{1}{\sqrt{3}} = -0.803\text{ and } y = -2.08\text{ (accept } -2.07\text{)}\]  
**A1**(**A1**)

The point of inflexion is \((-0.803, -2.08)\) \(\text{or } \left(-\frac{1}{\sqrt{3}}, -\frac{5}{3}\sqrt{3}\right)\)  
**C5**(**C1**)

**METHOD 2**
\[ f'(x) = 4x^3 - \frac{2}{x^2} \]  
(M1)(A1)

\[ f'(x) \text{ has a maximum when } x = -0.803 \]  
(M1)(A2)  
\[ y = -2.08 \text{ (accept } -2.07) \]  
(A1)  
(C1)  
[6]

40. (a) \[ 1300 = 650e^{20k} \]  
(M1)(A1)

\[ k = \frac{\ln 2}{20} \]  
(A1)  
(C3)

(b) \[ \frac{dn}{dt} = 650ke^{kt} \]  
(M1)(A1)

when \( t = 90 \), \[ \frac{dn}{dt} = 509.734 = 510 \text{ to } 3 \text{ sf} \]  
(A1)  
(C3)  
[6]

41. METHOD 1

(a) \[ f'(x) = \frac{(x+2)(2x+5)-(x^2+5x+5)}{(x+2)^2} \]  
(M1)(A1)

\[ = \frac{x^2 + 4x + 5}{(x+2)^2} \]  
(A1)  
(C3)

(b) \[ \frac{x^2 + 4x + 5}{(x+2)^2} > 2 \]

\[ \Rightarrow x^2 + 4x + 5 > 2x^2 + 8x + 8 \]  
(M1)(A1)

\[ \Rightarrow x^2 + 4x + 3 < 0 \]

\[ \Rightarrow -3 < x < -1 \]  
(A1)  
(C3)

METHOD 2

(a) \[ f(x) = x + 3 - \frac{1}{x+2} \]  
(A1)

\[ f'(x) = 1 + \frac{1}{(x+2)^2} \]  
(M1)(A1)

\[ (C3) \]

(b) \[ 1 + \frac{1}{(x+2)^2} > 2 \]

\[ (x+2)^2 < 1 \]  
(M1)

\[ -1 < x + 2 < 1 \]  
(A1)

\[ -3 < x < -1 \]  
(A1)  
(C3)  
[6]

42. (a) (i) \[ f'(x) = pe^{px}(x+1) + e^{px} \]  
(A1)

\[ = e^{px} \left( p(x+1) + 1 \right) \]  
(A1)

(ii) The result is true for \( n = 1 \) since

LHS = \( e^{px} \left( p(x+1) + 1 \right) \)

and RHS = \( p^{k+1}e^{px}(p(x+1) + 1) \) = \( e^{px}(p(x+1) + 1) \).

(M1)

Assume true for \( n = k \) : \( f^{(k)}(x) = p^{k-1}e^{px}(p(x+1) + k) \)  
(M1)

\[ f^{(k+1)}(x) = f^{(k)}(x)' = p^{k-1}pe^{px}(p(x+1) + k) + p^{k-1}e^{px}p \]  
(M1)(A1)

\[ = p^k e^{px}(p(x+1) + k + 1) \]  
(A1)

Therefore, true for \( n = k \Rightarrow \text{true for } n = k + 1 \) and the
The proposition is proved by induction. (R1) 7

(b) (i) \( f'(x) = e^{\sqrt[3]{x}} \left( \sqrt[3]{x} + 1 \right) = 0 \)

\[ \Rightarrow x = -\frac{1 + \sqrt[3]{3}}{\sqrt[3]{3}} = -\frac{\sqrt[3]{3} + 3}{3} \] (A1) N1

(ii) \( f''(x) = \sqrt[3]{3} e^{\sqrt[3]{x}} \left( \frac{\sqrt[3]{x} + 2}{\sqrt[3]{3}} \right) = 0 \)

\[ \Rightarrow x = -\frac{2 + \sqrt[3]{3}}{\sqrt[3]{3}} = -\frac{2\sqrt[3]{3} + 3}{3} \] (A1) N1 4

(c) \( f(x) = e^{0.5x} (x+1) \)

**EITHER**

area \( = -\int_{-1}^{1} f(x) \, dx + \int_{-1}^{2} f(x) \, dx \) (M1)

\[ = 8.08 \] (A1) N2

**OR**

area \( = \int_{-1}^{2} |f(x)| \, dx \) (M1)

\[ = 8.08 \] (A1) N2 2

[13]

43.

![Diagram of a circle with labeled points O, A, B, C, and N.](image)

(a) \( h = r \sin \theta \quad \text{CB = 2CN = 2r cos} \theta \) (A1)(A1)

Using \( T = (r + CB) \frac{h}{2} \) (M1)

\[ T = \frac{r^2}{2} \left( \sin \theta + 2 \sin \theta \cos \theta \right) \] (A1)

\[ = \frac{r^2}{2} \left( \sin \theta + \sin 2\theta \right) \] (AG) N0 4

(b) \( \frac{dT}{d\theta} = \frac{r^2}{2} \left( \cos \theta + 2 \cos 2\theta \right) = 0 \) (for max) (M1)

\[ \Rightarrow \cos \theta + 2(\cos^2 \theta - 1) = 4 \cos^2 \theta + \cos \theta - 2 = 0 \] (M1)(AG)

\[ \Rightarrow \cos \theta = 0.5931 \quad (\theta = 0.9359) \] (A1)

\[ \frac{d^2T}{d\theta^2} = \frac{r^2}{2} (-\sin \theta - 4 \sin 2\theta) \] (M1)

\[ \theta = 0.9359 \quad \Rightarrow \frac{d^2T}{d\theta^2} = -2.313r^2 < 0 \]

\[ \Rightarrow \text{there is a maximum (when } \theta = 0.9359 \text{)} \] (R1) 5

(c) In triangle AOB: \( AB = 2r \sin \frac{\theta}{2} \) (M1)(A1)

Perimeter OABC \( = 2r + 2r \cos \theta + 2r \sin \frac{\theta}{2} = 75 \) (M1)
When $\theta = 0.9359$, $r = 18.35 \text{ cm}$

Area $OABC = \frac{r^2}{2} (\sin \theta + \sin 2\theta) = \frac{18.35^2}{2} (\sin 0.9359 + \sin 1.872)$

$= 296 \text{ cm}^2$

(A1) N3 6

44. $f(x) = ax^3 + bx^2 + cx + d$

$\frac{d}{dx} f(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$f(0) = 2 = d$

$f'(1) = f(1) \to a + b + c + 2 = 3a + 2b + c$

$2 = 2a + b$

$f''(0) = -3 = c$

$f''(-1) = 6 = -6a + 2b$

$b = \frac{12}{5}, a = -\frac{1}{5}$

$A1A1$

$f(x) = -\frac{1}{5}x^3 + \frac{12}{5}x^2 - 3x + 2$ (Accept $a = -\frac{1}{5}, b = \frac{12}{5}, c = -3, d = 2$)

(C6)

45. (a) $r^{th}$ term $= \left( \frac{n}{n-r} \right) x^r h^{n-r}$

\[ = \frac{n!}{r!(n-r)!} x^r h^{n-r} \]

(A1)

(b) \[
\frac{d(x^n)}{dx} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}
\]

\[ = \lim_{h \to 0} \left\{ \frac{x^n + \binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \ldots + x^n}{h} \right\} \]

(A1)

\[ = \lim_{h \to 0} \left\{ \frac{x^n + nx^{n-1} h + \frac{n(n-1)}{2} x^{n-2} h^2 + \ldots + x^n}{h} \right\} \]

(A1)

\[ = \lim_{h \to 0} \left\{ nx^{n-1} + \frac{n(n-1)}{2} x^{n-2} h + \ldots + h^{n-1} \right\} \]

(A1)

\[ = nx^{n-1} \]

(A1)

Note: Accept first, second and last terms in the 3 lines above.

(c) $x^n \times x^{-n} = 1$

\[ x^n \frac{d(x^{-n})}{dx} + x^{-n} \frac{d(x^n)}{dx} = 0 \]

(M1)

\[ x^n \frac{d(x^{-n})}{dx} + x^{-n} \times nx^{n-1} = 0 \]

(A1)

\[ x^n \frac{d(x^{-n})}{dx} + nx^{n-1} = 0 \]

(A1)
\[
\frac{d(x^{-n})}{dx} = -nx^{-(1+n)}
\]  
(A1)

46. (a) \(f'(x) = 3 \cos^2(4x + 1) \times (-\sin(4x + 1)) \times 4\)  
\(f'(x) = -12 \cos^2(4x + 1) \sin(4x + 1)\)  
\text{Note: Award A1 for } 3 \cos^2(4x + 1), \text{ A1 for } -\sin(4x + 1) \text{ and A1 for 4.}  
(b) \(f'(x) = 0 \Rightarrow \cos^2(4x + 1) = 0 \text{ or } \sin(4x + 1) = 0\)  
\(\Rightarrow x = \frac{\pi}{8}, x = \frac{3\pi}{8}, x = \frac{\pi - 1}{4} \text{ or } x = \frac{\pi + 1}{4}\)  
\text{Note: Do not penalize the inclusion of additional answers.}  

47. \((\ln 3)^{3x+y} \left(1 + \frac{dy}{dx}\right) = 3x^2 + 3 \frac{dy}{dx}\)  
\(1\)  
\text{Note: Award A1 for } (\ln 3)^{3x+y}, \text{ A1 for } \left(1 + \frac{dy}{dx}\right) \text{ and A1 for } 3x^2 + 3 \frac{dy}{dx}.\)  
\(\frac{dy}{dx} \left((\ln 3)^{3x+y} - 3\right) = 3x^2 \times (\ln 3)^{-3x+y}\)  
\(\frac{dy}{dx} = \frac{3x^2 - (\ln 3)^{-3x+y}}{(\ln 3)^{3x+y} - 3}\)  
M1  
A1 N0  

48. (a) \(f'(x) = \frac{1}{3x+1} \times 3 \left(= \frac{3}{3x+1}\right)\)  
M1A1 N2  
(b) Hence when \(x = 2\), gradient of tangent = \(\frac{3}{7}\)  
\(\Rightarrow \text{gradient of normal is } -\frac{7}{3}\)  
\(y - \ln 7 = -\frac{7}{3}(x - 2)\)  
\(y = -\frac{7}{3}x + \frac{14}{3} + \ln 7\)  
\(\text{(accept } y = -2.33x + 6.61)\)  
A1 N4  

49. (a) \(\frac{dy}{dx} = -\sin \theta + i \cos \theta\)  
\(\text{EITHER}\)  
\(\frac{dy}{d\theta} = -i^2 \sin \theta + i \cos \theta\)  
\(= i (\cos \theta + i \sin \theta)\)  
\(= iy\)  
A1  
A1  
AG N0
OR

\[ y = i(\cos \theta + i \sin \theta) = i \cos \theta + i^2 \sin \theta \]
\[ = i \cos \theta - \sin \theta \]
\[ \frac{dy}{d\theta} \]

(b) \[ \int \frac{dy}{y} = i \int d\theta \]  
\[ \ln y = i \theta + c \]
\[ \text{Substituting } (0, 1) \quad 0 = 0 + c \Rightarrow c = 0 \]
\[ \Rightarrow \ln y = i \theta \]
\[ y = e^{i\theta} \]

(c) \[ \cos n\theta + i \sin n\theta = e^{in\theta} \]
\[ = (e^{i\theta})^n \]
\[ = (\cos \theta + i \sin \theta)^n \]

Note: Accept this proof in reverse.

(d) (i) \[ \cos 6\theta + i \sin 6\theta = (\cos \theta + i \sin \theta)^6 \]

Expanding RHS using the binomial theorem
\[ = \cos^6 \theta + 6 \cos^5 \theta i \sin \theta + 15 \cos^4 \theta (i \sin \theta)^2 + 20 \cos^3 \theta (i \sin \theta)^3 \]
\[ + 15 \cos^2 \theta (i \sin \theta)^4 + 6 \cos \theta (i \sin \theta)^5 + (i \sin \theta)^6 \]

Equating imaginary parts
\[ \sin 6\theta = 6 \cos^5 \theta \sin \theta - 20 \cos^3 \theta \sin^3 \theta + 6 \cos \theta \sin^5 \theta \]
\[ \sin 6\theta - \sin \theta = 6 \cos^5 \theta \sin \theta - 20 \cos^3 \theta \sin^3 \theta + 6 \cos \theta \sin^5 \theta \]
\[ = 32 \cos^5 \theta - 32 \cos^3 \theta + 6 \cos \theta (a = 32, b = -32, c = 6) \]

(ii) \[ \lim_{\theta \to 0} \frac{\sin 6\theta}{\sin \theta} = \lim_{\theta \to 0} \left(32 \cos^5 \theta - 32 \cos^3 \theta + 6 \cos \theta \right) \]
\[ = 32 - 32 + 6 \]
\[ = 6 \]

\[ \text{Note: Do not penalize the absence of } + C. \]

50. (a) \[ f'(x) = \ln x + \frac{1}{x} \]
\[ = \ln x \]  
(M1)  

(b) Using integration by parts

\[ \text{METHOD 1} \]
\[ \int (\ln x)^2 \, dx = (\ln x)^2 \cdot x - \int x \cdot 2(\ln x) \, dx \]
\[ = x(\ln x)^2 - 2 \int (\ln x) \, dx \]  
(A1)
\[ = x(\ln x)^2 - 2(x \ln x - x) + C \]
\[ = x(\ln x)^2 - 2x \ln x + x + C \]  
(A1)

\[ \text{METHOD 2} \]
\[ \int (\ln x)^2 \, dx = x(\ln x)^2 - x \ln x - \int (\ln x - 1) \, dx \]
\[ = x(\ln x)^2 - x \ln x - (x \ln x - x - x) + C \]
\[ = x(\ln x)^2 - 2x \ln x + x + C \]  
(A1)

\[ \text{Note: Do not penalize the absence of } + C. \]

51. \[ y = x \arcsin x \]
\[
\frac{dy}{dx} = \arcsin x + \frac{x}{\sqrt{1-x^2}}
\]

\[
\frac{d}{dx} \left( \frac{x}{\sqrt{1-x^2}} \right) = \frac{1}{1-x^2} \left( \frac{\sqrt{1-x^2}}{1-x^2} \right) + x^2 \left( 1-x^2 \right)^{\frac{1}{2}}
\]

\[
\frac{d^2 y}{dx^2} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\left(1-x^2\right)^{\frac{3}{2}}} + \frac{x^2}{\left(1-x^2\right)^{\frac{3}{2}}}
\]

\[
= \frac{2}{\left(1-x^2\right)^{\frac{3}{2}}}
\]

\[
= \frac{2 - x^2}{\left(1-x^2\right)^{\frac{3}{2}}}
\]

**Note:** The final A1A1 are for equivalent algebraic manipulations leading to AG.

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52. \( e^{xy} - y^2 \ln x = 1 \)

Differentiating implicitly

\[
e^{xy} \left( y + x \frac{dy}{dx} \right) - \left( 2y \frac{dy}{dx} \ln x + \frac{y^2}{x} \right) = 0
\]

**Notes:** Award A1 for \( e^{xy} \left( y + x \frac{dy}{dx} \right), \) A1 for

\[
- \left( 2y \frac{dy}{dx} \ln x + \frac{y^2}{x} \right) \quad \text{and} \quad 0.
\]

**N.B.** Incorrect manipulation of \( e^{xy} \) can lead to the correct final result.

**EITHER**

Collecting terms \( e^{xy} \left( y - \frac{y^2}{x} \right) = \left( 2y \ln x - xe^{xy} \right) \frac{dy}{dx} \)

\[
\frac{dy}{dx} = \frac{ye^{xy} - \frac{y^2}{x}}{2y \ln x - xe^{xy}}
\]

\[
x = 1 \quad y = 1 \quad \frac{dy}{dx} = \frac{1 - e}{e}
\]

**OR**

Substituting \( x = 1 \) \( y = 1 \)
53. (a) derivative = \( \frac{3f'(x)[g(x)-1]-3f(x)g'(x)}{[g(x)-1]^2} \)  
when \( x = 0 \) \( \Rightarrow \) derivative = \( \frac{3(1)(-4-1)-3(1)(5)}{(-4-1)^2} \)  
\( = -3 \)  
(\text{A1})  

(b) derivative = \( f'(g(x) + 2x)(g'(x) + 2) \)  
when \( x = 1 \) derivative = \( f'(-1 + 2)(2 + 2) \)  
\( = (3)(4) \)  
\( = 12 \)  
(\text{A1})  

54. (a) Use of quotient (or product) rule  
\( \frac{\text{d}}{\text{d}x} = \frac{2(x^2 + 6) - (2x \times 2x)}{(x^2 + 6)^2} \)  
\( = \frac{12 - 2x^2}{(x^2 + 6)^2} \)  
(\text{M1})  

(b) Solving \( f'(x) = 0 \) for \( x \)  
\( x = \pm \sqrt{6} \)  
(\text{M1})  

\( f \) has to be \( 1 - 1 \) for \( f^{-1} \) to exist and so the least value of \( b \)  
is the larger of the two \( x \)-coordinates  
(accept a labelled sketch)  
(\text{R1})  
Hence \( b = \sqrt{6} \)  
(\text{A1})  

55. (a) Attempting implicit differentiation  
\( 2x + y + x \frac{\text{d}y}{\text{d}x} + 2y \frac{\text{d}y}{\text{d}x} = 0 \)  
(\text{M1})  

\text{EITHER}  
Substituting \( x = -1, y = k \) \( \Rightarrow -2 + k \frac{\text{d}y}{\text{d}x} + 2k \frac{\text{d}y}{\text{d}x} = 0 \)  
(\text{M1})  

Attempting to make \( \frac{\text{d}y}{\text{d}x} \) the subject  
(\text{M1})  

\text{OR}  
Substituting \( x = -1, y = k \) into \( \frac{\text{d}y}{\text{d}x} \)  
(\text{M1})  

\text{THEN}  
\( \frac{\text{d}y}{\text{d}x} = \frac{2 - k}{2k - 1} \)  
(\text{A1})  

(b) Solving \( \frac{\text{d}y}{\text{d}x} = 0 \) for \( k \) gives \( k = 2 \)  
(\text{A1})
56. (a) \( AQ = \sqrt{x^2 + 4} \) (km) (A1)  
\( QY = (2 - x) \) (km) (A1)  
\( T = 5\sqrt{5} \cdot AQ + 5\cdot QY \) (M1)  
\( = 5\sqrt{5} \cdot (x^2 + 4) + 5(2 - x) \) (mins) A1

(b) Attempting to use the chain rule on \( 5\sqrt{5} \cdot \sqrt{x^2 + 4} \) (M1)  
\[ \frac{d}{dx} \left( 5\sqrt{5} \cdot \sqrt{x^2 + 4} \right) = 5\sqrt{5} \cdot \frac{1}{2} \cdot x^2 + 4 \cdot \frac{1}{2} \cdot 2x \] A1
\[ = \frac{5\sqrt{5} \cdot x}{\sqrt{x^2 + 4}} \]  
\[ \frac{d}{dx} (5 - x) = -5 \] A1  
\[ \frac{dT}{dx} = \frac{5\sqrt{5} \cdot x}{\sqrt{x^2 + 4}} - 5 \] AG N0

(c) (i) \( \sqrt{5} \cdot x = \sqrt{x^2 + 4} \) or equivalent A1  
Squaring both sides and rearranging to obtain \( 5x^2 = x^2 + 4 \) M1  
\( x = 1 \) A1 N1  
*Note:* Do not award the final A1 for stating a negative solution in final answer.

(ii) \( T = 5\sqrt{5} \cdot \sqrt{1^2 + 4} + 5(2 - 1) \) M1  
\( = 30 \) (mins) A1 N1  
*Note:* Allow FT on incorrect \( x \) value.

(iii) **METHOD 1**  
Attempting to use the quotient rule M1  
\[ u = x, \quad v = \sqrt{x^2 + 4}, \quad \frac{du}{dx} = 1 \quad \text{and} \quad \frac{dv}{dx} = x(x^2 + 4)^{-1/2} \] (A1)  
\[ \frac{d^2T}{dx^2} = 5\sqrt{5} \cdot \left[ \frac{\sqrt{x^2 + 4} - \frac{1}{2} \cdot (x^2 + 4)^{-1/2} \cdot 2x^2}{(x^2 + 4)} \right] \] A1  
Attempt to simplify (M1)  
\[ = \frac{5\sqrt{5}}{(x^2 + 4)^{3/2}} \cdot [x^2 + 4 - x^2] \] or equivalent A1  
\[ = \frac{20\sqrt{5}}{(x^2 + 4)^{3/2}} \] AG  
When \( x = 1 \), \( \frac{20\sqrt{5}}{(x^2 + 4)^{3/2}} > 0 \) and hence \( T = 30 \) is a minimum R1 N0  
*Note:* Allow FT on incorrect \( x \) value, \( 0 \leq x \leq 2 \).

**METHOD 2**  
Attempting to use the product rule M1
\[ u = x, \quad v = \sqrt{x^2 + 4}, \quad \frac{du}{dx} = 1 \quad \text{and} \quad \frac{dv}{dx} = x\left(x^2 + 4\right)^{-1/2} \]  
\[ \frac{d^2T}{dx^2} = 5\sqrt{5}\left(x^2 + 4\right)^{-1/2} - \frac{5\sqrt{5}x}{2}\left(x^2 + 4\right)^{-3/2} \times 2x \]
\[ \left( \frac{5\sqrt{5}}{(x^2 + 4)^{3/2}} - \frac{5\sqrt{5}x^2}{(x^2 + 4)^{3/2}} \right) \]
\[ \text{Attempt to simplify} \]
\[ = \frac{5\sqrt{5}(x^2 + 4) - 5\sqrt{5}x^2}{(x^2 + 4)^{3/2}} \left( \frac{5\sqrt{5}(x^2 + 4 - x^2)}{(x^2 + 4)^{3/2}} \right) \]
\[ = \frac{20\sqrt{5}}{(x^2 + 4)^{3/2}} \]

When \( x = 1 \), \( \frac{20\sqrt{5}}{(x^2 + 4)^{3/2}} > 0 \) and hence \( T = 30 \) is a minimum.

**Note:** Allow FT on incorrect \( x \) value, \( 0 \leq x \leq 2 \).  

57. **METHOD 1**
\[ 3x^2y^2 + 2x^3y \frac{dy}{dx} = -\pi \sin \left(\frac{\pi y}{2}\right) \frac{dy}{dx} \]
\[ \text{At } (-1, 1), \quad 3 - 2 \frac{dy}{dx} = 0 \]
\[ \frac{dy}{dx} = \frac{3}{2} \]

**METHOD 2**
\[ 3x^2y^2 + 2x^3y \frac{dy}{dx} = -\pi \sin \left(\frac{\pi y}{2}\right) \frac{dy}{dx} \]
\[ \frac{dy}{dx} = \frac{3x^2y^2}{-\pi \sin \left(\frac{\pi y}{2}\right) - 2x^3y} \]
\[ \text{At } (-1, 1), \quad \frac{dy}{dx} = \frac{3(-1)^2(1)^2}{-\pi \sin \left(\frac{\pi}{2}\right) - 2(-1)^3(1)} = \frac{3}{2} \]