IB Practice - Calculus - Differentiation (V2 Legacy)

1. If $2x^2 - 3y^2 = 2$, find the two values of $\frac{dy}{dx}$ when $x = 5$.

Working:	
	Answer: ………………………………………………..

(Total 4 marks)

2. Differentiate $y = \arccos(1 - 2x^2)$ with respect to x, and simplify your answer.

Working:	
	Answer: ………………………………………………..

(Total 4 marks)

3. **Give exact answers in this part of the question.**
The temperature $g(t)$ at time t of a given point of a heated iron rod is given by

$$g(t) = \frac{\ln t}{\sqrt{t}}, \qquad \text{where } t > 0.$$

(a) Find the interval where $g'(t) > 0$.
(4)

(b) Find the interval where $g''(t) > 0$ and the interval where $g''(t) < 0$.
(5)

(c) Find the value of t where the graph of $g(t)$ has a point of inflexion.
(3)

(d) Let t^* be a value of t for which $g'(t^*) = 0$ and $g''(t^*) < 0$. Find t^*.
(3)

(e) Find the point where the normal to the graph of $g(t)$ at the point $(t^*, g(t^*))$ meets the t-axis.
(3)

(Total 18 marks)

4. Let $f(x) = \ln|x^5 - 3x^2|$, $-0{\cdot}5 < x < 2$, $x \neq a$, $x \neq b$; (a, b are values of x for which $f(x)$ is not defined).

(a) (i) Sketch the graph of $f(x)$, indicating on your sketch the number of zeros of $f(x)$. Show also the position of any asymptotes.
(2)

(ii) Find all the zeros of $f(x)$, (that is, solve $f(x) = 0$).
(3)

(b) Find the **exact** values of a and b.
(3)

(c) Find $f(x)$, and indicate clearly where $f'(x)$ is not defined.
(3)

(d) Find the **exact** value of the x-coordinate of the local maximum of $f(x)$, for $0 < x < 1.5$. (You may assume that there is no point of inflexion.)
(3)

(e) **Write down** the definite integral that represents the area of the region **enclosed** by $f(x)$ and the x-axis. (Do **not** evaluate the integral.)
(2)

(Total 16 marks)

5. Differentiate from first principles $f(x) = \cos x$.

(Total 8 marks)

6. For the function $f : x \mapsto x^2 \ln x$, $x > 0$, find the function f', the derivative of f with respect to x.

(Total 3 marks)

7. For the function $f : x \mapsto \frac{1}{2}\sin 2x + \cos x$, find the possible values of $\sin x$ for which $f'(x) = 0$.

Working:

Answer:

....

(Total 3 marks)

8. For what values of m is the line $y = mx + 5$ a tangent to the parabola $y = 4 - x^2$?

Working:

Answer:

....

(Total 3 marks)

9. The tangent to the curve $y^2 - x^3$ at the point P(1, 1) meets the x-axis at Q and the y-axis at R.
Find the ratio PQ : QR.

Working:

Answer:

....

(Total 3 marks)

10. (a) Sketch and label the curves

$y = x^2$ for $-2 \le x \le 2$, and $y = -\frac{1}{2}\ln x$ for $0 < x \le 2$.

(2)

(b) Find the x-coordinate of P, the point of intersection of the two curves.

(2)

(c) If the tangents to the curves at P meet the y-axis at Q and R, calculate the area of the triangle PQR.

(6)

(d) Prove that the two tangents at the points where $x = a$, $a > 0$, on each curve are always perpendicular.

(4)

(Total 14 marks)

11. (a) Let $y = \frac{a + b\sin x}{b + a\sin x}$, where $0 < a < b$.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(b^2 + a^2)\cos x}{(b + a\sin x)^2}$.

(4)

(ii) Find the maximum and minimum values of y.

(4)

(iii) Show that the graph of $y = \frac{a + b\sin x}{b + a\sin x}$, $0 < a < b$ cannot have a vertical asymptote.

(2)

(b) For the graph of $y = \frac{4 + 5\sin x}{5 + 4\sin x}$ for $0 \le x \le 2\pi$,

(i) write down the y-intercept;
(ii) find the x-intercepts m and n, (where $m < n$) correct to four significant figures;
(iii) sketch the graph.

(5)

(c) The area enclosed by the graph of $y = \frac{4 + 5\sin x}{5 + 4\sin x}$ and the x-axis from $x = 0$ to $x = n$ is denoted by A. Write down, but do **not** evaluate, an expression for the area A.

(2)

(Total 17 marks)

12. If $f(x) = \ln(2x - 1)$, $x > \frac{1}{2}$, find

(a) $f'(x)$;

(b) the value of x where the gradient of $f(x)$ is equal to x.

Working:

Answers:
(a) ....
(b) ....

(Total 3 marks)

13. Find the x-coordinate, between –2 and 0, of the point of inflexion on the graph of the function $f : x \mapsto x^2 e^x$. Give your answer to 3 decimal places.

Working:

Answer:
…………………………………………..

(Total 3 marks)

14. Find the gradient of the tangent to the curve $3x^2 + 4y^2 = 7$ at the point where x = 1 and $y > 0$.

Working:

Answer:
....

(Total 3 marks)

15. The function f is given by $f : x \mapsto e^{(1+\sin \pi x)}$, $x \geq 0$.

(a) Find $f'(x)$.

Let x_n be the value of x where the $(n + 1)^{\text{th}}$ maximum or minimum point occurs, $n \in \mathbb{N}$. (*ie* x_0 is the value of x where the first maximum or minimum occurs, x_1 is the value of x where the second maximum or minimum occurs, *etc*).

(b) Find x_n in terms of n.

Working:

Answers:
(a) ....
(b) ....

(Total 3 marks)

16. Let $f(x) = x\left(\sqrt[3]{(x^2 - 1)^2}\right)$, $-1.4 \leq x \leq 1.4$

(a) **Sketch** the graph of $f(x)$. (An exact scale diagram is **not** required.)
On your graph indicate the approximate position of
(i) each zero;
(ii) each maximum point;
(iii) each minimum point.

(4)

(b) (i) Find $f'(x)$, clearly stating its domain.
(ii) Find the x-coordinates of the maximum and minimum points of $f(x)$, for $-1 < x < 1$.

(7)

(c) Find the x-coordinate of the point of inflexion of $f(x)$, where $x > 0$, giving your answer correct to **four** decimal places.

(2)

(Total 13 marks)

17. The line $y = 16x - 9$ is a tangent to the curve $y = 2x^3 + ax^2 + bx - 9$ at the point (1,7). Find the values of a and b.

(Total 3 marks)

18. Consider the function $y = \tan x - 8 \sin x$.

(a) Find $\frac{dy}{dx}$.

(b) Find the value of $\cos x$ for which $\frac{dy}{dx} = 0$.

Working:

Answers:
(a) ....
(b) ....

(Total 3 marks)

19. Consider the tangent to the curve $y = x^3 + 4x^2 + x - 6$.

(a) Find the equation of this tangent at the point where $x = -1$.

(b) Find the coordinates of the point where this tangent meets the curve again.

Working:

Answers:
(a) ....
(b) ....

(Total 3 marks)

20. Let $y = \sin(kx) - kx\cos(kx)$, where k is a constant.

Show that $\frac{dy}{dx} = k^2 x \sin(kx)$.

(Total 3 marks)

21. Consider the function $f(x) = x^{\frac{1}{x}}$, where $x \in \mathbb{R}^+$.

(a) Show that the derivative $f'(x) = f(x)\left(\frac{1 - \ln x}{x^2}\right)$.

(3)

(b) Sketch the function $f(x)$, showing clearly the local maximum of the function and its horizontal asymptote. You may use the fact that

$$\lim_{x \to \infty} \frac{\ln x}{x} = 0.$$

(5)

(c) Find the Taylor expansion of $f(x)$ about $x = e$, up to the second degree term, and show that this polynomial has the same maximum value as $f(x)$ itself.

(5)

(Total 13 marks)

22. A curve has equation $xy^3 + 2x^2y = 3$. Find the equation of the tangent to this curve at the point (1, 1).

Working:

Answer:
....

(Total 6 marks)

23. The function f is defined by

$$f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}$$

(a) (i) Find an expression for $f'(x)$, simplifying your answer.

(ii) The tangents to the curve of $f(x)$ at points A and B are parallel to the x-axis. Find the coordinates of A and of B.

(5)

(b) (i) Sketch the graph of $y = f'(x)$.

(ii) Find the x-coordinates of the three points of inflexion on the graph of f.

(5)

(c) Find the range of

(i) f;

(ii) the composite function $f \circ f$.

(5)

(Total 15 marks)

24. Air is pumped into a spherical ball which expands at a rate of 8 cm^3 per second (8 cm^3 s^{-1}). Find the **exact** rate of increase of the radius of the ball when the radius is 2 cm.

Working:

Answer: ....

(Total 6 marks)

25. A curve has equation $x^3 y^2 = 8$. Find the equation of the normal to the curve at the point (2, 1).

Working:

Answer: ....

(Total 6 marks)

26. The function f is defined by $f(x) = \frac{x^2}{2^x}$, for $x > 0$.

(a) (i) Show that

$$f'(x) = \frac{2x - x^2 \ln 2}{2^x}$$

(ii) Obtain an expression for $f''(x)$, simplifying your answer as far as possible.

(5)

(b) (i) Find the **exact** value of x satisfying the equation $f'(x) = 0$

(ii) Show that this value gives a maximum value for $f(x)$.

(4)

(c) Find the x-coordinates of the two points of inflexion on the graph of f.

(3)

(Total 12 marks)

27. Consider the function $f(t) = 3\sec^2 t + 5t$.

(a) Find $f'(t)$.

(b) Find the **exact** values of

(i) $f(\pi)$;

(ii) $f'(\pi)$;

Working:

Answers:

(a) ....

(b) (i) ....

(ii) ....

(Total 6 marks)

28. Consider the equation $2xy^2 = x^2y + 3$.

(a) Find y when $x = 1$ and $y < 0$.

(b) Find $\frac{dy}{dx}$ when $x = 1$ and $y < 0$.

Working:

Answers:

(a) ....

(b) ....

(Total 6 marks)

29. Let $y = e^{3x} \sin(\pi x)$.

(a) Find $\frac{dy}{dx}$.

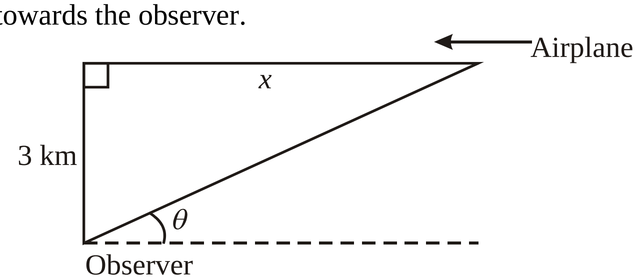
(b) Find the smallest positive value of x for which $\frac{dy}{dx} = 0$.

Working:

Answers:
(a) ....
(b) ....

(Total 6 marks)

30. An airplane is flying at a constant speed at a constant altitude of 3 km in a straight line that will take it directly over an observer at ground level. At a given instant the observer notes that the angle θ is $\frac{1}{3}\pi$ radians and is increasing at $\frac{1}{60}$ radians per second. Find the speed, in kilometres per hour, at which the airplane is moving towards the observer.



Working:

Answer:
....

(Total 6 marks)

31. A curve has equation $f(x) = \frac{a}{b + e^{-cx}}$, $a \neq 0$, $b > 0$, $c > 0$.

(a) Show that $f''(x) = \frac{ac^2 e^{-cx}(e^{-cx} - b)}{(b + e^{-cx})^3}$.

(4)

(b) Find the coordinates of the point on the curve where $f''(x) = 0$.

(2)

(c) Show that this is a point of inflexion.

(2)

(Total 8 marks)

32. The point P(1, p), where $p > 0$, lies on the curve $2x^2y + 3y^2 = 16$.

(a) Calculate the value of p.

(b) Calculate the gradient of the tangent to the curve at P.

Working:

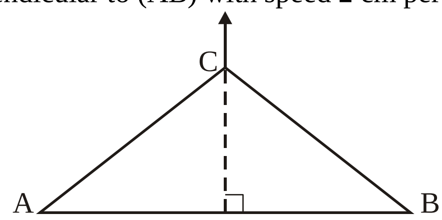
Answers:
(a) ....
(b) ....

(Total 6 marks)

33. The function f is defined by $f : x \mapsto 3^x$.
Find the solution of the equation $f''(x) = 2$.

(Total 6 marks)

34. The following diagram shows an isosceles triangle ABC with AB = 10 cm and AC = BC. The vertex C is moving in a direction perpendicular to (AB) with speed 2 cm per second.



Calculate the rate of increase of the angle $\hat{CAB}$ at the moment the triangle is equilateral.

Working:

Answer:
....

(Total 6 marks)

35. If $y = \ln(2x - 1)$, find $\frac{d^2y}{dx^2}$.

Working:

Answer:
....

(Total 6 marks)

36. Find the equation of the normal to the curve $x^3 + y^3 - 9xy = 0$ at the point (2, 4).

Working:

Answer:
....

(Total 6 marks)

37. The function f' is given by $f'(x) = 2\sin\left(5x - \frac{\pi}{2}\right)$.

(a) Write down $f''(x)$.

(b) Given that $f\left(\frac{\pi}{2}\right) = 1$, find $f(x)$.

....
....

(Total 6 marks)

38. Find the gradient of the normal to the curve $3x^2y + 2xy^2 = 2$ at the point (1, –2).

....
....
....
....

(Total 6 marks)

39. The function f is given by $f(x) = \frac{x^5 + 2}{x}$, $x \neq 0$. There is a point of inflexion on the graph of f at the point P. Find the coordinates of P.

(Total 6 marks)

40. An experiment is carried out in which the number n of bacteria in a liquid, is given by the formula $n = 650\, e^{kt}$, where t is the time in minutes after the beginning of the experiment and k is a constant. The number of bacteria doubles every 20 minutes. Find

(a) the **exact** value of k;

(b) the rate at which the number of bacteria is increasing when $t = 90$.

(Total 6 marks)

41. Let $f(x) = \dfrac{x^2 + 5x + 5}{x + 2}$, $x \neq -2$.

(a) Find $f'(x)$.

(b) Solve $f'(x) > 2$.

(Total 6 marks)

42. The function f is defined by $f(x) = e^{px}(x + 1)$, here $p \in \mathbb{R}$.

(a) (i) Show that $f'(x) = e^{px}(p(x + 1) + 1)$.

(ii) Let $f^{(n)}(x)$ denote the result of differentiating $f(x)$ with respect to x, n times. Use mathematical induction to prove that

$$f^{(n)}(x) = p^{n-1}e^{px}(p(x + 1) + n),\ n \in \mathbb{Z}^+.$$

(7)

(b) When $p = \sqrt{3}$, there is a minimum point and a point of inflexion on the graph of f. Find the **exact** value of the x-coordinate of

(i) the minimum point;

(ii) the point of inflexion.

(4)

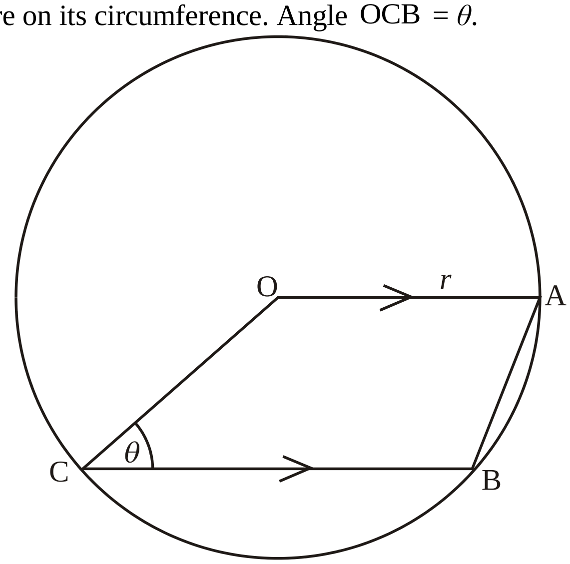
(c) Let $p = \dfrac{1}{2}$. Let R be the region enclosed by the curve, the x-axis and the lines $x = -2$ and $x = 2$.

Find the area of R.

(2)

(Total 13 marks)

43. The diagram shows a trapezium OABC in which OA is parallel to CB. O is the centre of a circle radius r cm. A, B and C are on its circumference. Angle $\text{O}\hat{\text{C}}\text{B} = \theta$.



Let T denote the area of the trapezium OABC.

(a) Show that $T = \dfrac{r^2}{2}(\sin\theta + \sin 2\theta)$.

(4)

For a **fixed** value of r, the value of T varies as the value of θ varies.

(b) Show that T takes its maximum value when θ satisfies the equation $4\cos^2\theta + \cos\theta - 2 = 0$, and verify that this value of T is a maximum.

(5)

(c) Given that the perimeter of the trapezium is 75 cm, find the maximum value of T.

(6)

(Total 15 marks)

44. Let f be a cubic polynomial function. Given that $f(0) = 2$, $f'(0) = -3$, $f(1) = f'(1)$ and $f''(-1) = 6$, find $f(x)$.

(Total 6 marks)

45. (a) Write down the term in x^r in the expansion of $(x + h)^n$, where $0 \le r \le n$, $n \in \mathbb{Z}^+$.

(1)

(b) Hence differentiate x^n, $n \in \mathbb{Z}^+$, from first principles.

(5)

(c) Starting from the result $x^n \times x^{-n} = 1$, deduce the derivative of x^{-n}, $n \in \mathbb{Z}^+$.

(4)

(Total 10 marks)

46. Let $f(x) = \cos^3(4x + 1)$, $0 \le x \le 1$.

(a) Find $f'(x)$.

(b) Find the **exact** values of the three roots of $f'(x) = 0$.

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....

(Total 6 marks)

47. Given that $3^{x+y} = x^3 + 3y$, find $\frac{dy}{dx}$.

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....

....

(Total 6 marks)

48. Let f be the function defined for $x > -\frac{1}{3}$ by $f(x) = \ln(3x + 1)$.

(a) Find $f'(x)$.

(b) Find the equation of the normal to the curve $y = f(x)$ at the point where $x = 2$.
Give your answer in the form $y = ax + b$ where $a, b \in \mathbb{R}$.

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(Total 6 marks)

49. Let $y = \cos\theta + i\sin\theta$.

(a) Show that $\frac{dy}{d\theta} = iy$.

[You may assume that for the purposes of differentiation and integration, i may be treated in the same way as a real constant.]

(3)

(b) **Hence** show, using integration, that $y = e^{i\theta}$.

(5)

(c) Use this result to deduce de Moivre's theorem.

(2)

(d) (i) Given that $\frac{\sin 6\theta}{\sin\theta} = a\cos^5\theta + b\cos^3\theta + c\cos\theta$, where $\sin\theta \ne 0$, use de Moivre's theorem with $n = 6$ to find the values of the constants a, b and c.

(ii) Hence deduce the value of $\lim_{\theta \to 0} \frac{\sin 6\theta}{\sin\theta}$.

(10)

(Total 20 marks)

50. Let $f(x) = x \ln x - x$, $x > 0$.

(a) Find $f'(x)$.

(b) Using integration by parts find $\int (\ln x)^2 \, dx$.

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....

....

(Total 6 marks)

51. Let $y = x \arcsin x$, $x \in \,]-1, 1[$. Show that $\frac{d^2 y}{dx^2} = \frac{2 - x^2}{(1 - x^2)^{\frac{3}{2}}}$.

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(Total 6 marks)

52. Given that $e^{xy} - y^2 \ln x = e$ for $x \geq 1$, find $\frac{dy}{dx}$ at the point (1, 1).

....

....

....

(Total 6 marks)

53. The following table shows the values of two functions f and g and their first derivatives when $x = 1$ and $x = 0$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	4	1	–4	5
1	–2	3	–1	2

(a) Find the derivative of $\frac{3f(x)}{g(x) - 1}$ when $x = 0$.

(b) Find the derivative of $f(g(x) + 2x)$ when $x = 1$.

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....

(Total 6 marks)

54. The function f is defined by $f(x) = \frac{2x}{x^2 + 6}$ for $x \geq b$ where $b \in \mathbb{R}$.

(a) Show that $f'(x) = \frac{12 - 2x^2}{(x^2 + 6)^2}$.

(b) Hence find the smallest **exact** value of b for which the inverse function f^{-1} exists. Justify your answer.

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(Total 6 marks)

55. Consider the curve with equation $x^2 + xy + y^2 = 3$.

(a) Find in terms of k, the gradient of the curve at the point $(-1, k)$.

(5)

(b) Given that the tangent to the curve is parallel to the x-axis at this point, find the value of k.

(1)

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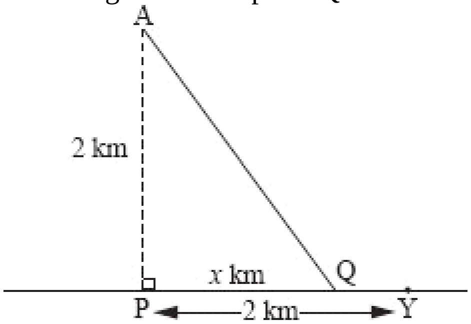
....

(Total 6 marks)

56. André wants to get from point A located in the sea to point Y located on a straight stretch of beach. P is the point on the beach nearest to A such that AP = 2 km and PY = 2 km. He does this by swimming in a straight line to a point Q located on the beach and then running to Y.



When André swims he covers 1 km in $5\sqrt{5}$ minutes. When he runs he covers 1 km in 5 minutes.

(a) If PQ = x km, $0 \le x \le 2$, find an expression for the time T minutes taken by André to reach point Y.

(4)

(b) Show that $\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{5\sqrt{5}\,x}{\sqrt{x^2+4}} - 5$.

(3)

(c) (i) Solve $\frac{\mathrm{d}T}{\mathrm{d}x} = 0$.

(ii) Use the value of x found in **part (c) (i)** to determine the time, T minutes, taken for André to reach point Y.

(iii) Show that $\frac{\mathrm{d}^2T}{\mathrm{d}x^2} = \frac{20\sqrt{5}}{(x^2+4)^{\frac{3}{2}}}$ and **hence** show that the time found in **part (c) (ii)** is a minimum.

(11)

(Total 18 marks)

57. Find the gradient of the tangent to the curve $x^3 y^2 = \cos(\pi y)$ at the point $(-1, 1)$.

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(Total 12 marks)

IB Practice - Calculus - Differentiation: (V2 Legacy) MarkScheme

1. By implicit differentiation,

$\frac{d}{dx}(2x^2 - 3y^2 = 2) \Rightarrow 4x - 6y\frac{dy}{dx} = 0$ (M1)

$\Leftrightarrow \frac{dy}{dx} = \frac{4x}{6y}$ (A1)

When $x = 5$, $50 - 3y^2 = 2$ (M1)

$\Leftrightarrow y^2 = 16$

$\Leftrightarrow y = \pm 4$

Therefore $\frac{dy}{dx} = \pm\frac{5}{6}$ (A1)

(C2)(C2)

Note: *This can be done explicitly.*

[4]

2. Given $y = \arccos(1 - 2x^2)$

then $\frac{dy}{dx} = \frac{-1}{(1-(1-2x^2)^2)^{1/2}} \times -4x$ (M1)

$\frac{dy}{dx} = \frac{4x}{(1-(1-4x^2+4x^4))^{1/2}}$ (M1)

$\frac{dy}{dx} = \frac{4x}{(4x^2-4x^4)^{1/2}}$ (A2)

OR

$\cos y = 1 - 2x^2$ (M1)

$-\sin y\frac{dy}{dx} = -4x$

$\frac{dy}{dx} = \frac{-4x}{-\sin y} = \frac{4x}{\sqrt{1-(1-2x^2)^2}}$ (M1)

$\frac{dy}{dx} = \frac{4x}{\sqrt{4x^2-4x^4}}$ (A2)

(C4)

[4]

3. (a) $g(t) = \frac{\ln t}{\sqrt{t}}$. So $g'(t) = \frac{2-\ln t}{2t^{3/2}}$. (M1)(A1)

Hence, $g'(t) > 0$ when $2 > \ln t$ or $\ln t < 2$ or $t < e^2$. (M1)

Since the domain of $g(t)$ is $\{t{:}t > 0\}$, $g'(t) > 0$ when $0 < t < e^2$. (A1) 4

(b) Since $g'(t) = \frac{2-\ln t}{2t^{3/2}}$, $g''(t) = \frac{-2\sqrt{t} - 3\sqrt{t}(2-\ln t)}{4t^3}$ (M2)

$= -\frac{\sqrt{t}[8-3\ln t]}{4t^3}$ (A1)

Hence $g''(t) > 0$ when $8 - 3\ln t < 0$ *ie* $t > e^{8/3}$. (M1)

Similarly, $g''(t) < 0$ when $0 < t < e^{8/3}$. (A1) 5

(c) $g''(t) = 0$ when $t = 0$ or $8 = 3\ln t$.

Since, the domain of g is $\{t{:}t > 0\}$, $g''(t) = 0$ when $t = e^{8/3}$. (M1)

Since $g''(t) > 0$ when $t > e^{8/3}$ and $g''(t) < 0$ when $t < e^{8/3}$, (M1)

$\left(e^{8/3}, \frac{8}{3}e^{-4/3}\right)$ is the point of inflexion. The required value of t is $e^{8/3}$. (A1) 3

Note: *Award (A1) for evaluating t as* $e^{8/3}$.

(d) $g'(t) = 0$ when $\ln t = 2$ or $t = e^2$. (M1)

Also $g''(e^2) = -\frac{\sqrt{e^2}[8 - 3\ln e^2]}{4e^6} = -\frac{1}{2e^5} < 0$ (M1)

Hence $t^* = e^2$ (A1) 3

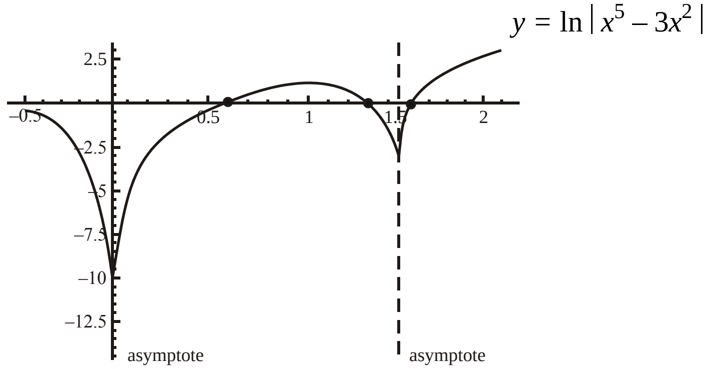
(e) At (t^*, $g(t^*)$) the tangent is horizontal. (M1)

So the normal at the point (t^*, $g(t^*)$) is the line $t = t^*$. (M1)

Thus, it meets the t axis at the point $t = t^* = e^2$ and hence the point is $(e^2, 0)$. (A1) 3

[18]

4. (a) (i)



(G2)

Note: *Award (G1) for correct shape, including three zeros, and (G1) for both asymptotes*

(ii) $f(x) = 0$ for $x = 0.599, 1.35, 1.51$ (G1)(G1)(G1) 5

(b) $f(x)$ is undefined for

$(x^5 - 3x^2) = 0$ (M1)

$x^2(x^3 - 3) = 0$

Therefore, $x = 0$ or $x = 3^{1/3}$ (A2) 3

(c) $f'(x) = \frac{5x^4 - 6x}{x^5 - 3x^2}\left(\text{or } \frac{5x^3 - 6}{x^4 - 3x}\right)$ (M1)(A1)

$f'(x)$ is undefined at $x = 0$ and $x = 3^{1/3}$ (A1) 3

(d) For the x-coordinate of the local maximum of $f(x)$, where

$0 < x < 1.5$ put $f'(x) = 0$ (R1)

$5x^3 - 6 = 0$ (M1)

$x = \left(\frac{6}{5}\right)^{\frac{1}{3}}$ (A1) 3

(e) The required area is

$A = \int_{0.599}^{1.35} f(x)\,dx$ (A2) 2

Note: *Award (A1) for each correct limit.*

[16]

5. Using first principles

$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$

$= \lim_{h\to 0}\left(\frac{\cos(x+h)-\cos(x)}{h}\right)$ (M1)

$= \lim_{h\to 0}\left(\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}\right)$ (M1)(A1)

$= \cos(x)\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right)-\sin(x)\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right)$ (M1)(A1)

But $\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right) = 1$ and $\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right) = 0$ (C1)(C1)

Therefore, $f'(x) = -\sin x$ (A1)

OR

$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$

$= \lim_{h\to 0}\left(\frac{\cos(x+h)-\cos(x)}{h}\right)$ (M1)

$= \lim_{h\to 0}\left(\frac{-2\sin(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h}\right)$ (using any method) (M1)(A2)

$= \lim_{h\to 0}\left(-\sin\left(x+\frac{1}{2}h\right)\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\right)$ (M1)

But $= \lim_{h\to 0}\left(\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\right) = 1$ and $= \lim_{h\to 0}\left(-\sin\left(x+\frac{1}{2}h\right)\right) = -\sin x$ (C2)

Therefore, $f'(x) = -\sin x$ (A1) 8

[8]

6. $f(x) = x^2 \ln x$

$f'(x) = 2x \ln x + x^2\left(\frac{1}{x}\right)$ (M1)(M1)

$= 2x \ln x + x$ (A1)

(C3)

$f' : x \mapsto 2x \ln x + x$

[3]

7. $f(x) = \frac{1}{2}\sin 2x + \cos x$

$f'(x) = \cos 2x - \sin x$ (M1)

$= 1 - 2\sin^2 x - \sin x$

$= (1 + \sin x)(1 - 2\sin x)$ (M1)

$= 0$ when $\sin x = -1$ or $\frac{1}{2}$ (A1)

(C3)

[3]

8. **Method 1:** $y = 4 - x^2$

$\frac{dy}{dx} = -2x = m$ when $x = -\frac{m}{2}$ (M1)

Thus, $\left(-\frac{m}{2}, 4 - \frac{m^2}{4}\right)$ lies on $y = mx + 5$. (R1)

Then, $4 - \frac{m^2}{4} = -\frac{m^2}{2} + 5$, so $m^2 = 4$

$m = \pm 2$. (A1)

(C3)

Method 2: For intersection: $mx + 5 = 4 - x^2$ or $x^2 + mx + 1 = 0$. (M1)

For tangency: discriminant = 0 (M1)

Thus, $m^2 - 4 = 0$

$m = \pm 2$ (A1)

(C3)

[3]

9. $y^2 = x^3$ so $2y\frac{dy}{dx} = 3x^2$.

At P(1, 1), $\frac{dy}{dx} = \frac{3}{2}$. (M1)

The tangent is $3x - 2y = 1$, giving Q = $\left(\frac{1}{3}, 0\right)$ and R = $\left(0, \frac{-1}{2}\right)$. (A1)

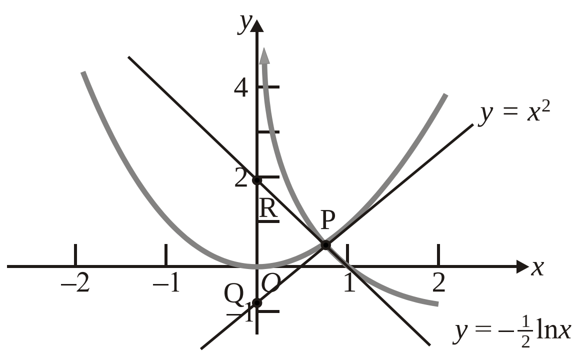
Therefore, PQ : QR = $\frac{2}{3} : \frac{1}{3}$ or $1 : \frac{1}{2}$

= 2 : 1. (A1)

(C3)

[3]

10. (a)



(C2) 2

Note: *Award (C1) for $y = x^2$, (C1) for $y = -\frac{1}{2}\ln x$.*

(b) $x^2 + \frac{1}{2}\ln x = 0$ when $x = 0.548217$.

Therefore, the x-coordinate of P is 0.548…. (G2) 2

(c) The tangent at P to $y = x^2$ has equation $y = 1.0964x - 0.30054$, (G2)

and the tangent at P to $y = -\frac{1}{2}\ln x$ has equation $y = -0.91205x + 0.80054$. (G2)

Thus, the area of triangle PQR = $\frac{1}{2}(0.30052 + 0.80054)(0.5482)$. (M1)

= 0.302 (3 sf) (A1)

OR

$y = x^2 \Rightarrow \frac{dy}{dx} = 2x$ (M1)

Therefore, the tangent at (p, p^2) has equation $2px - y = p^2$. (C1)

$y = -\frac{1}{2}\ln x \Rightarrow \frac{dy}{dx} = -\frac{1}{2x}$ (M1)

Therefore, the tangent at (p, p^2) has equation $x + 2py = p + 2p^3$. (C1)

Thus, $Q = (0, -p^2)$ and $R = (0, p^2 + \frac{1}{2})$.

Thus, the area of the triangle PQR

$= \frac{1}{2}(2p^2 + \frac{1}{2})p$ (M1)

$= 0.302$ (3 sf) (A1) 6

(d) $y = x^2 \Rightarrow$ when $x = a$, $\frac{dy}{dx} = 2a$ (C1)

$y = -\frac{1}{2}\ln x \Rightarrow$ when $x = a$, $\frac{dy}{dx} = -\frac{1}{2a}$ $(a > 0)$ (C1)

Now, $(2a)\left(-\frac{1}{2a}\right) = -1$ for all $a > 0$. (M1)

Therefore, the tangents to the curve at $x = a$ on each curve are always perpendicular. (R1)(AG) 4

[14]

11. (a) (i) $y = \frac{a + b\sin x}{b + a\sin x}$, $0 < a < b$

$$\frac{dy}{dx} = \frac{(b + a\sin x)(b\cos x) - (a + b\sin x)(a\cos x)}{(b + a\sin x)^2}$$ (M1)(C1)

$$= \frac{b^2\cos x + ab\sin x\cos x - a^2\cos x - ab\sin x\cos x}{(b + a\sin x)^2}$$ (M1)(C1)

$$= \frac{(b^2 - a^2)\cos x}{(b + a\sin x)^2}$$ (AG) 4

(ii) $\frac{dy}{dx} = 0 \Rightarrow \cos x = 0$ since $b^2 - a^2 \neq 0$.

This gives $x = \frac{\pi}{2}$ $(+\pi k, k \in \mathbb{Z})$ (M1)(C1)

When $x = \frac{\pi}{2}$, $y = \frac{a + b}{b + a} = 1$,

and when $x = \frac{3\pi}{2}$, $y = \frac{a - b}{b - a} = -1$.

Therefore, maximum $y = 1$ and minimum $y = -1$. (A2) 4

(iii) A vertical asymptote at the point x exists if and only if $b + a\sin x = 0$. (R1)

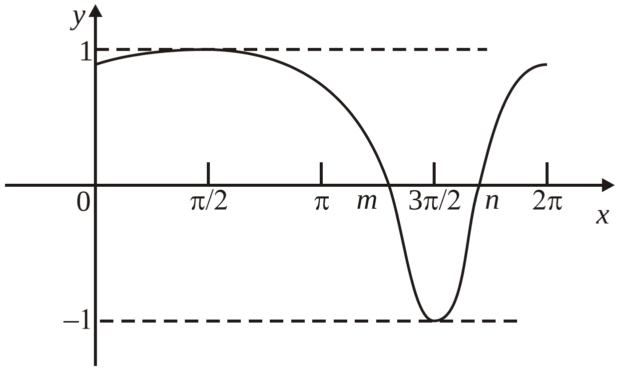
Then, since $0 < a < b$, $\sin x = -\frac{b}{a} < -1$, which is impossible. (R1)

Therefore, no vertical asymptote exists. (AG) 2

(b) (i) y-intercept $= 0.8$ (A1)

(ii) For x-intercepts, $\sin x = -\frac{4}{5} \Rightarrow x = 4.069, 5.356$. (A2)

(iii)



(C2) 5

(c) Area = $\int_0^{4.069} \frac{4+5\sin x}{5+4\sin x}dx - \int_{4.069}^{5.356} \frac{4+5\sin x}{5+4\sin x}dx$ (M1)(C1)

OR

Area = $\int_0^{5.356} \left|\frac{4+5\sin x}{5+4\sin x}\right|dx$ (M1)(C1) 2

[17]

12. (a) If $f(x) = \ln(2x - 1)$,

Then $f'(x) = \frac{2}{2x-1}$ (A2)

(b) Put $\frac{2}{2x-1} = x$

$\Rightarrow x - 1.28$ (using a graphic display calculator or the quadratic formula) (A1)

[3]

13. If $f : (x) \mapsto x^2e^x$

then $f'(x) = x^2e^x + 2xe^x$

$f''(x) = x^2e^x + 4xe^x + 2e^x = e^x(x^2 + 4x + 2)$ (A1)

For a point of inflexion solve $f''(x) = 0$

$f''(x) = 0$ at $x = -0.586$ (using a graphic display calculator or the quadratic formula) (A1)

(Since $f'(x) \neq 0$ at this value, then it is a point of inflexion.)

Note: *Some candidates may find the value of x from f′(x) by finding the minimum turning point using a graphic display calculator*

[3]

14. $3x^2 + 4y^2 = 7$

When $x = 1$, $y = 1$ (since $y > 0$) (M1)

$\frac{d}{dx}(3x^2 + 4y^2 = 7) \Rightarrow 6x + 8y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{3x}{4y}$ (A1)

The gradient where $x = 1$ and $y = 1$ is $-\frac{3}{4}$ (A1)(C3)

OR

$3x^2 + 4y^2 = 7$

$\Rightarrow y = \sqrt{\frac{7-3x^2}{4}}$, since $y > 0$ (M1)

(A1)

$\frac{dy}{dx} = -\frac{3x}{2(7-3x)^{\frac{1}{2}}}$, (A1)

$= -\frac{3}{4}$, when $x = 1$ (A1)(C3)

[3]

15. (a) $f'(x) = \pi \cos(\pi x)e^{(1+\sin \pi x)}$ (A1) (C1)

(b) For maximum or minimum points, $f'(x) = 0$

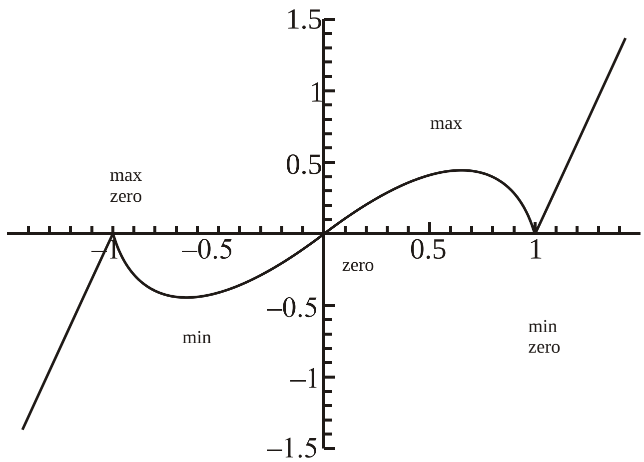
$\cos \pi x = 0$ (M1)

$\pi x = \frac{2k+1}{2}\pi$

then $x_n = \frac{2n+1}{2}$ (A1)(C2)

[3]

16. (a) $f(x) = x\left(\sqrt[3]{(x^2-1)^2}\right)$



(A4) 4

Notes: *Award (A1) for the shape, including the two cusps (sharp points) at x = ±1.*
(i) Award (A1) for the zeros at x = ±1 and x = 0.
(ii) Award (A1) for the maximum at x = –1 and the minimum at x = 1.
(iii) Award (A1) for the maximum at approx. x = 0.65 , and the minimum at approx. x = –0.65
There are no asymptotes.
The candidates are not required to draw a scale.

(b) (i) Let $f(x) = x(x^2-1)^{\frac{2}{3}}$

Then $f'(x) = \frac{4}{3}x^2(x^2-1)^{-\frac{1}{3}} + (x^2-1)^{\frac{2}{3}}$ (M1)(A2)

$f'(x) = (x^2-1)^{-\frac{1}{3}}\left[\frac{4}{3}x^2 + (x^2-1)\right]$

$f'(x) = (x^2-1)^{-\frac{1}{3}}\left(\frac{7}{3}x^2 - 1\right)$ (or equivalent)

$f'(x) = \frac{7\boldsymbol{x}^2 - 3}{3(\boldsymbol{x}^2-1)^{\frac{1}{3}}}$ (or equivalent)

The domain is $-1.4 \le x \le 1.4$, $x \ne \pm 1$ (accept $-1.4 < x < 1.4$, $x \ne \pm 1$) (A1)

(ii) For the maximum or minimum points let $f'(x) = 0$

ie $(7x^2 - 3) = 0$ or use the graph. (M1)

Therefore, the x-coordinate of the maximum point is

$x = \sqrt{\frac{3}{7}}$ (or 0.655) and (A1)

the x-coordinate of the minimum point is $x = -\sqrt{\frac{3}{7}}$ (or –0.655). (A1) 7

Notes: *Candidates may do this using a GDC, in that case award (M1)(G2).*

(c) The x-coordinate of the point of inflexion is $x = \pm 1.1339$ (G2)

OR

$f''(x) = \frac{4x(7x^2 - 9)}{9\sqrt[3]{(x^2-1)^4}}, x \neq \pm 1$ (M1)

For the points of inflexion let $f''(x) = 0$ and use the graph,

ie $x = \sqrt{\frac{9}{7}} = 1.1339.$ (A1) 2

Note: *Candidates may do this by plotting $f'(x)$ and finding the x-coordinate of the minimum point. There are other possible methods.*

[13]

17. For the curve, $y = 7$ when $x = 1 \Rightarrow a + b = 14$, and (M1)

$\frac{dy}{dx} = 6x^2 + 2ax + b = 16$ when $x = 1 \Rightarrow 2a + b = 10.$ (M1)

Solving gives $a = -4$ and $b = 18$. (A1)(C3)

[3]

18. (a) $\frac{dy}{dx} = \sec^2 x - 8\cos x$ (A1) (C1)

(b) $\frac{dy}{dx} = \frac{1 - 8\cos^3 x}{\cos^2 x}$ (M1)

$\frac{dy}{dx} = 0$

$\Rightarrow \cos x = \frac{1}{2}$ (A1)(C2)

[3]

19. **METHOD 1**

(a) The equation of the tangent is $y = -4x - 8$. (G2)(C2)

(b) The point where the tangent meets the curve again is (–2, 0). (G1)(C1)

METHOD 2

(a) $y = -4$ and $\frac{dy}{dx} = 3x^2 + 8x + 1 = -4$ at $x = -1$. (M1)

Therefore, the tangent equation is $y = -4x - 8$. (A1)(C2)

(b) This tangent meets the curve when $-4x - 8 = x^3 + 4x^2 + x - 6$ which gives

$x^3 + 4x^2 + 5x + 2 = 0 \Rightarrow (x + 1)^2(x + 2) = 0$.

The required point of intersection is (–2, 0). (A1)(C1)

[3]

20. $y = \sin(kx) - kx\cos(kx)$

$\frac{dy}{dx} = k\cos(kx) - k\{\cos(kx) + x[-k\sin(kx)]\}$ (M1)(C1)

$= k\cos(kx) - k\cos(kx) + k^2 x\sin(kx)$ (C1)

$= k^2 x\sin(kx)$ (AG)

[3]

21. (a) The derivative can be found by logarithmic differentiation. Let $y = f(x)$.

$y = x^{\frac{1}{x}} \Rightarrow \ln y = \frac{1}{x}\ln x$ (M1)

$\Rightarrow \frac{y'}{y} = \frac{-1}{x^2}\ln x + \frac{1}{x}\times\frac{1}{x} = \frac{1-\ln x}{x^2}$ (M1)(M1)

$\Rightarrow y' = y\left(\frac{1-\ln x}{x^2}\right)$

that is, $f'(x) = f(x)\left(\frac{1-\ln x}{x^2}\right)$ (AG) 3

(b) This function is defined for positive and real numbers only.
To find the exact value of the local maximum:

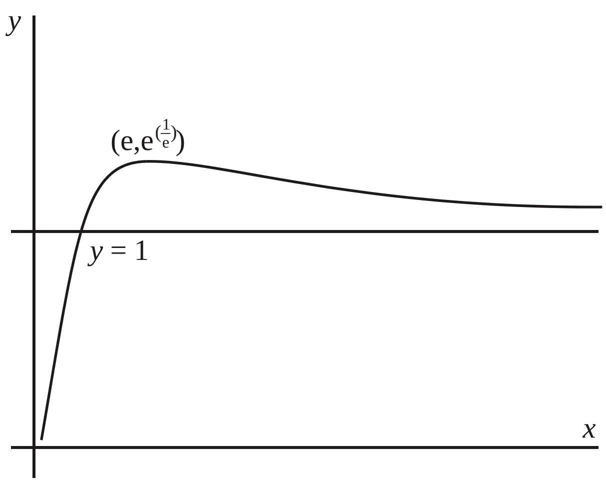
$y' = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$ (M1)

$\Rightarrow y = e^{\frac{1}{e}}$ (A1)

To find the horizontal asymptote:

$\lim_{x\to\infty}(y = x^{\frac{1}{x}}) \Rightarrow \lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln x}{x} = 0$

$\Rightarrow \lim_{x\to\infty} y = 1$ (M1)(A1)



(A1) 5

(c) By Taylor's theorem we have

$P_2(x) = f(e) + f'(e)(x - e) + \frac{f''(e)}{2}(x - e)^2$ (A1)

$f''(x) = f'(x)\left(\frac{1-\ln x}{x^2}\right) + f(x)\left(\frac{2\ln x - 3}{x^3}\right)$ (M1)

Also, $f'(e) = 0$, and $f''(e) = 0 + f(e)\left(\frac{2-3}{e^3}\right) = e^{\frac{1}{e}}\left(\frac{-1}{e^3}\right) = -e^{\frac{1}{e}-3}$ (M1)(A1)

hence $P_2(x) = e^{\frac{1}{e}} - \frac{e^{\frac{1}{e}-3}}{2}(x - e)^2$ which is a parabola with vertex

at $x = e$ and $P_2(e) = e^{\frac{1}{e}} = f(e)$ (R1)(AG) 5

[13]

22. $y^3 + 3xy^2\frac{dy}{dx} + 4xy + 2x^2\frac{dy}{dx} = 0$ (M1)(A1)

$\Rightarrow \dfrac{dy}{dx} = \dfrac{-(y^3+4xy)}{3xy^2+2x^2}$ (A1)

At (1,1), $\dfrac{dy}{dx} = -1$ (A1)

Equation of tangent is $y - 1 = -1(x - 1)$ or $x + y = 2$ (A2) (C6)

[6]

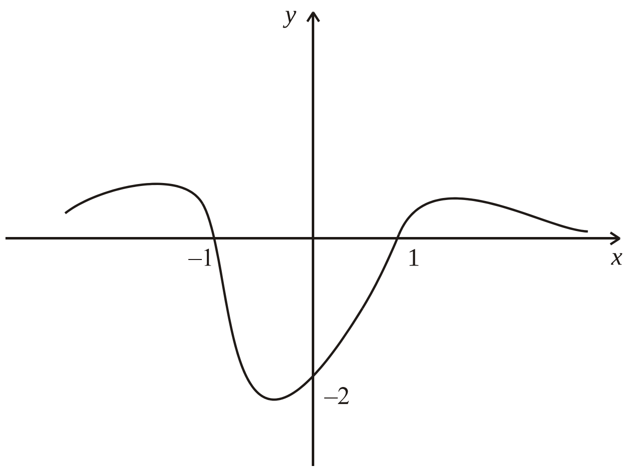
23. (a) (i) $f'(x) = \dfrac{(2x-1)(x^2+x+1)-(2x+1)(x^2-x+1)}{(x^2+x+1)^2}$ (M1)(A1)

$= \dfrac{2(x^2-1)}{(x^2+x+1)^2}$ (A1)

(ii) $f'(x) = 0 \Rightarrow x = \pm 1$

$A\left(1, \frac{1}{3}\right)$ B(–1, 3) $\left(\text{or } A(-1, 3)\ B\left(1, \frac{1}{3}\right)\right)$ (A1)(A1) 5

(b) (i)



(G2)

Note: *Award (G1) for general shape and (G1) for indication of scale.*

(ii) The points of inflexion can be found by locating the max/min on the graph of f'.

This gives $x = -1.53, -0.347, 1.88$. (G3)

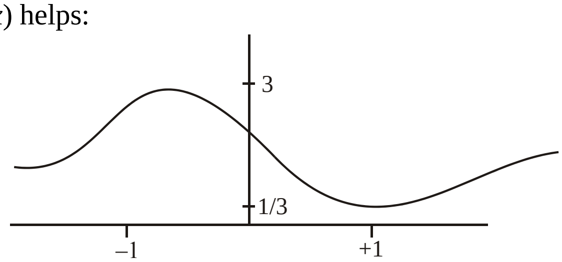
OR

$f''(x) = \dfrac{-4(x^3-3x-1)}{(x^2+x+1)^3}$ (M1)

$f''(x) = 0 \Rightarrow x^3 - 3x - 1 = 0$ (A1)

$\Rightarrow x = 1.53, -0.347, 1.88$ (G1) 5

(c) The graph of $y = f(x)$ helps:



(i) Range of f is $\left[\frac{1}{3}, 3\right]$. (A1)(A1)

(ii) We require the image set of $\left[\frac{1}{3}, 3\right]$.

$$f\left(\frac{1}{3}\right) = \frac{\frac{1}{9} - \frac{1}{3} + 1}{\frac{1}{9} + \frac{1}{3} + 1} = \frac{7}{13},\ f(3) = \frac{9-3+1}{9+3+1} = \frac{7}{13}$$ (M1)

Range of g is $\left[\frac{1}{3}, \frac{7}{13}\right]$. (A1)(A1) 5

Note: *Since the question did not specify **exact** ranges accept open intervals or numerical approximations (eg [0.333, 0.538]).*

[15]

24. $\frac{dV}{dt} = 8$ (cm^{3}s^{-1}), $V = \frac{4}{3}\pi r^3$

$\Rightarrow \frac{dV}{dr} = 4\pi r^2$ (M1)(A1)

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \left(\frac{dV}{dt}\right) \div \left(\frac{dV}{dr}\right)$ (M1)

When $r = 2$, $\frac{dr}{dt} = 8 \div (4\pi \times 2^2)$ (M1)(A1)

$= \frac{1}{2\pi}$ (cm s^{-1}) (do **not** accept 0.159) (A1) (C6)

[6]

25. METHOD 1

$3x^2y^2 + x^3 2y\frac{dy}{dx} = 0$ (M1)(A1)

At (2, 1), $12 + 16\frac{dy}{dx} = 0$ (M1)

$\Rightarrow \frac{dy}{dx} = -\frac{3}{4}$ (A1)

Gradient of normal $= \frac{4}{3}$ (A1)

Equation of normal is $y - 1 = \frac{4}{3}(x - 2)$ (A1) (C6)

METHOD 2

$y = 2\sqrt{2}x^{-\frac{3}{2}}$ (A1)

$\frac{dy}{dx} = -3\sqrt{2}x^{-\frac{5}{2}}$ (M1)(A1)

$= -\frac{3}{4}$ when $x = 2$ (A1)

Gradient of normal = $\frac{4}{3}$ (A1)

Equation of normal is $y - 1 = \frac{4}{3}(x - 2)$ (A1) (C6)

[6]

26. (a) (i) $f'(x) = \frac{2x \cdot 2^x - x^2 2^x \ln 2}{2^{2x}}$ (M1)(A1)

$= \frac{2x - x^2 \ln 2}{2^x}$ (AG)

(ii) $f''(x) = \frac{2^x[2 - 2x\ln 2] - 2^x \ln 2[2x - x^2 \ln 2]}{2^{2x}}$ (M1)(A1)

$= \frac{x^2(\ln 2)^2 - 4x\ln 2 + 2}{2^x}$ (A1) 5

Note: *Award the second (A1) for some form of simplification,*

eg accept $\frac{x\ln 2(x\ln 2 - 4) + 2}{2^x}$.

(b) (i) $2x - x^2 \ln 2 = 0$ giving $x = \frac{2}{\ln 2}$ (M1)(A1)

Note: *Award (M1)(A0) for $x = 2.89$.*

(ii) With this value of x,

$f''(x) = \frac{4 - 8 + 2}{+\text{ve number}} < 0$ (M1)(A1)

Therefore, a maximum. (AG) 4

(c) Points of inflexion satisfy $f''(0) = 0$, *ie*

$x^2 (\ln 2)^2 - 4x \ln 2 + 2 = 0$ (M1)

$\Rightarrow x = \frac{4\ln 2 \pm \sqrt{8(\ln 2)^2}}{2(\ln 2)^2}$ (A1)

$= \frac{2 \pm \sqrt{2}}{\ln 2}$ (= 0.845, 4.93) (A1)

OR

$x = 0.845, 4.93$

1) (M1)(G1)(G 3

[12]

27. (a) **METHOD 1**

$f(t) = 3\sec^2 t + 5t$

$f(t) = 3(\cos t)^{-2} + 5t$

$f(t) = -6(\cos t)^{-3}(-\sin t) + 5$ (M1)(A1)

$= \frac{6\sin t}{\cos^3 t} + 5$ (C2)

METHOD 2

$f'(t) = 3 \times 2\sec t(\sec t \tan t) + 5$ (M1)(A1)

$= 6\sec^2 t \tan t + 5 (= 6\tan^2 t + 6\tan t + 5)$

(C2)

(b) $f(\pi) = \dfrac{3}{(\cos\pi)^2} + 5\pi$ (M1)

$= 3 + 5\pi$ (A1)

(C2)

$f'(\pi) = \dfrac{6\sin\pi}{(\cos\pi)^3} + 5$ (M1)

$= 5$ (A1)

(C2)

[6]

28. $2xy^2 = x^2y + 3$

(a) $x = 1 \Rightarrow 2y^2 - y - 3 = 0$ (M1)

$y = \dfrac{3}{2}$ or $y = -1$

$y < 0 \Rightarrow y = -1$ (A1)

(C2)

(b) $2y^2 + 4xy\dfrac{dy}{dx} = 2xy + x^2\dfrac{dy}{dx}$

(M1)(M1)(A1)

$\left(\dfrac{dy}{dx} = \dfrac{2xy - 2y^2}{4xy - x^2}\right)$

$(1, -1) \Rightarrow \dfrac{dy}{dx} = \dfrac{4}{5}$ (A1)

(C4)

[6]

29. $y = e^{3x}\sin(\pi x)$

(a) $\dfrac{dy}{dx} = 3e^{3x}\sin(\pi x) + \pi e^{3x}\cos(\pi x)$

(M1)(A1)(A1)

(C3)

(b) $0 = e^{3x}(3\sin(\pi x) + \pi\cos(\pi x))$

$\tan(\pi x) = -\dfrac{\pi}{3}$ (M1)

$\pi x = -0.80845 + \pi$ (M1)

$x = 0.7426\ldots$ (0.743 to 3 sf) (A1)

(C3)

[6]

30. $\tan\theta = \dfrac{3}{x}$

$\sec^2\theta\,\dfrac{d\theta}{dt} = \dfrac{-3}{x^2}\dfrac{dx}{dt}$ (M1)

when $\theta = \dfrac{\pi}{3}$, $x^2 = 3$ and $\sec^2\theta = 4$ (A1)(A1)

$\dfrac{dx}{dt} = \dfrac{-x^2\sec^2\theta}{3}\dfrac{d\theta}{dt}$ (M1)

$\dfrac{dx}{dt} = \dfrac{-3(4)}{3}\left(\dfrac{1}{60}\right)$

$\frac{dx}{dt} = -\frac{1}{15}$ km s^{-1}

$\frac{dx}{dt}$ = –240 km h^{-1} (A1)

The airplane is moving towards him at 240 km h^{-1} (A1)
(C6)

***Note:** Award (C5) if the answer is given as –240 km h^{-1}.*

[6]

31. $f(x) = \frac{a}{b + e^{-cx}}$, $a \neq 0$, $b > 0$, $c > 0$

(a) $f'(x) = \frac{(b+e^{-cx})(0) - (a)(-ce^{-cx})}{(b+e^{-cx})^2}$ (M1)

$= \frac{ace^{-cx}}{(b+e^{-cx})^2}$ (A1)

$f''(x) = \frac{(b+e^{-cx})^2(-ac^2e^{-cx}) - (ace^{-cx})2(b+e^{-cx})(-ce^{-cx})}{(b+e^{-cx})^4}$ (M1)

$= \frac{-bac^2e^{-cx} - ac^2(e^{-cx})^2 + 2ac^2(e^{-cx})^2}{(b+e^{-cx})^3}$

$= \frac{ac^2(e^{-cx})^2 - bac^2e^{-cx}}{(b+e^{-cx})^3}$ (A1)

$= \frac{ac^2e^{-cx}(e^{-cx} - b)}{(b+e^{-cx})^3}$ (AG) 4

(b) $f''(x) = 0 \Rightarrow e^{-cx} = b$

$\Rightarrow -cx = \ln b$

$\Rightarrow x = -\frac{1}{c}\ln b \Rightarrow y = \frac{a}{b + e^{\ln b}} = \frac{a}{2b}$ 2 a b =

So coordinate = $\left(-\frac{1}{c}\ln b, \frac{a}{2b}\right)$ (A1)(A1) 2

(c) Now $e^{-cx} > b$ on one side of $x = -\frac{1}{c}$ ln b and $e^{-cx} < b$ on the other side. (R1)

$\Rightarrow f''(x)$ changes sign at this point. (R1)

$\Rightarrow$ It is a point of inflexion. (AG) 2

[8]

32. (a) $2p + 3p^2 = 16 \Rightarrow p = 2$ (M1)(A1) (C2)

***Note:** Do **not** penalize if $p = -\frac{8}{3}$ also appears.*

(b) $4xy + 2x^2\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$ (A1)(A1)

***Note:** Award (A1) for $4xy + 2x^2\frac{dy}{dx}$ and (A1) for $6y\frac{dy}{dx} = 0$.*

at P(1, 2), $8 + 2\frac{dy}{dx} + 12\frac{dy}{dx} = 0 \Rightarrow 14\frac{dy}{dx} = -8$ (A1)

$\Rightarrow$ gradient $= -\frac{4}{7}$ (= –0.571) (A1)(C4)

[6]

33. $f(x) = 3^x \Rightarrow f'(x) = 3^x \ln 3$ (M1)

$\Rightarrow f''(x) = 3^x (\ln 3)^2$ (A1)

$3^x(\ln 3)^2 = 2$

$3^x = \dfrac{2}{(\ln 3)^2}$ (A1)

$x \ln 3 = \ln\left(\dfrac{2}{(\ln 3)^2}\right)$ (M1)

$x = \dfrac{\ln\left(\dfrac{2}{(\ln 3)^2}\right)}{\ln 3}$ (A1)

$= 0.460$ (A1)(C6)

[6]

34. Let h = height of triangle and $\theta = \hat{CAB}$. (A1)

Then, $h = 5 \tan \theta$ (A1)

$\dfrac{dh}{dt} = 5\sec^2\theta \times \dfrac{d\theta}{dt}$ (M1)(A1)

Put $\theta = \dfrac{\pi}{3}$.

$2 = 5 \times 4 \times \dfrac{d\theta}{dt}$ (A1)

$\dfrac{d\theta}{dt} = \dfrac{1}{10}$ rad per sec $\left(\text{Accept } \dfrac{18°}{\pi} \text{ per second or } 5.73° \text{ per second}\right)$ (A1)(A1)

(C6)

Note: *Award (A1) for the correct value, and (A1) for the correct units.*

[6]

35. $y = \ln(2x - 1)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{2}{2x-1}$ (M1)(A1)

$\Rightarrow \dfrac{dy}{dx} = 2(2x-1)^{-1}$ (A1)

$\Rightarrow \dfrac{d^2y}{dx^2} = -2(2x-1)^{-2}\,(2)$ (M1)(A1)

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{-4}{(2x-1)^2} \text{ or } -4(2x-1)^{-2}$ (A1)(C6)

[6]

36. $x^3 + y^3 - 9xy = 0$

Differentiating w.r.t. x

$\Rightarrow 3x^2 + 3y^2\dfrac{dy}{dx} - 9y - 9x\dfrac{dy}{dx} = 0$ (A1)(A1)

Note: *Award (A1) for* $3x^2 + 3y^2\dfrac{dy}{dx}$, *and (A1) for* $-9y - 9x\dfrac{dy}{dx}$.

$\Rightarrow \dfrac{dy}{dx} = \dfrac{9y - 3x^2}{3y^2 - 9x}$ (A1)

EITHER

at point (2, 4) gradient = 0.8. (A1)

$\Rightarrow$ Gradient of normal = –1.25 (A1)

OR

Gradient of normal $= \frac{-3y^2 + 9x}{9y - 3x^2}$ (A1)

at point (2, 4), gradient is –1.25 (A1)

THEN

Equation of normal is given by

$y - 4 = -1.25(x - 2)$ or $y = -1.25x + 6.5$ (A1)(C6)

[6]

37. (a) Using the chain rule $f'(x) = \left(2\cos\left(5x - \frac{\pi}{2}\right)\right)5$ (M1)

$= 10\cos\left(5x - \frac{\pi}{2}\right)$ A1 2

(b) $f(x) = \int f'(x)\,dx$

$= -\frac{2}{5}\cos\left(5x - \frac{\pi}{2}\right) + c$ A1

Substituting to find c, $f\left(\frac{\pi}{2}\right) = -\frac{2}{5}\cos\left(5\left(\frac{\pi}{2}\right) - \frac{\pi}{2}\right) + c = 1$ M1

$c = 1 + \frac{2}{5}\cos 2\pi = 1 + \frac{2}{5} = \frac{7}{5}$ (A1)

$f(x) = -\frac{2}{5}\cos\left(5x - \frac{\pi}{2}\right) + \frac{7}{5}$ A1 N2 4

[6]

38. Attempting to differentiate implicitly (M1)

$3x^2y + 2xy^2 = 2 \Rightarrow 6xy + 3x^2\frac{dy}{dx} + 2y^2 + 4xy\frac{dy}{dx} = 0$ A1

Substituting $x = 1$ and $y = -2$ (M1)

$-12 + 3\frac{dy}{dx} + 8 - 8\frac{dy}{dx} = 0$ A1

$\Rightarrow -5\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = -\frac{4}{5}$ A1

Gradient of normal is $\frac{5}{4}$. A1 N3

[6]

39. **METHOD 1**

$f'(x) = 4x^3 - \frac{2}{x^2}$ (M1)(A1)

$f''(x) = 12x^2 + \frac{4}{x^3}$ (A1)

$f''(x) = 0$ (M1)

$\Rightarrow x = -\frac{1}{\sqrt[5]{3}} = -0.803$ and $y = -2.08$ (accept -2.07) (A1)(A1)

The point of inflexion is $(-0.803, -2.08)$ $\left(\text{or } \left(-\frac{1}{\sqrt[5]{3}}, -\frac{5}{3}\sqrt[5]{3}\right)\right)$ (C5)(C1)

METHOD 2

$f'(x) = 4x^3 - \dfrac{2}{x^2}$ (M1)(A1)

$f'(x)$ has a maximum when $x = -0.803$ (M1)(A2) (C5)

$y = -2.08$ (accept -2.07) (A1) (C1)

[6]

40. (a) $1300 = 650e^{20k}$ (M1)(A1)

$k = \dfrac{\ln 2}{20}$ (A1) (C3)

(b) $\dfrac{dn}{dt} = 650ke^{kt}$ (M1)(A1)

when $t = 90$, $\dfrac{dn}{dt} = 509.734 = 510$ to 3 sf (A1) (C3)

[6]

41. **METHOD 1**

(a) $f'(x) = \dfrac{(x+2)(2x+5)-(x^2+5x+5)}{(x+2)^2}$ (M1)(A1)

$= \dfrac{x^2+4x+5}{(x+2)^2}$ (A1) (C3)

(b) $\dfrac{x^2+4x+5}{(x+2)^2} > 2$

$\Rightarrow x^2+4x+5 > 2x^2+8x+8$ (M1)(A1)

$\Rightarrow x^2+4x+3 < 0$

$\Rightarrow -3 < x < -1$ (A1) (C3)

METHOD 2

(a) $f(x) = x+3-\dfrac{1}{x+2}$ (A1)

$f'(x) = 1+\dfrac{1}{(x+2)^2}$ (M1)(A1)

(C3)

(b) $1+\dfrac{1}{(x+2)^2} > 2$

$(x+2)^2 < 1$ (M1)

$-1 < x+2 < 1$ (A1)

$-3 < x < -1$ (A1) (C3)

[6]

42. (a) (i) $f'(x) = pe^{px}(x+1)+e^{px}$ (A1)

$= e^{px}\left(p(x+1)+1\right)$ (AG)

(ii) The result is true for $n = 1$ since

$\text{LHS} = e^{px}\left(p(x+1)+1\right)$

and $\text{RHS} = p^{1-1}e^{px}\left(p(x+1)+1\right) = e^{px}\left(p(x+1)+1\right)$. (M1)

Assume true for $n = k$: $f^{(k)}(x) = p^{k-1}e^{px}\left(p(x+1)+k\right)$ (M1)

$f^{(k+1)}(x) = \left(f^{(k)}(x)\right)' = p^{k-1}pe^{px}\left(p(x+1)+k\right)+p^{k-1}e^{px}p$ (M1)(A1)

$= p^k e^{px}\left(p(x+1)+k+1\right)$ (A1)

Therefore, true for $n = k \Rightarrow$ true for $n = k+1$ and the proposition is proved by induction. (R1) 7

(b) (i) $f'(x) = e^{\sqrt{3}x}\left(\sqrt{3}(x+1)+1\right) = 0$ (M1)

$\Rightarrow x = -\dfrac{1+\sqrt{3}}{\sqrt{3}} \left(= -\dfrac{\sqrt{3}+3}{3}\right)$ (A1) N1

(ii) $f''(x) = \sqrt{3}e^{\sqrt{3}x}\left(\sqrt{3}(x+1)+2\right) = 0$ (M1)

$\Rightarrow x = -\dfrac{2+\sqrt{3}}{\sqrt{3}} \left(= -\dfrac{2\sqrt{3}+3}{3}\right)$ (A1)N1 4

(c) $f(x) = e^{0.5x}(x+1)$

EITHER

area $= -\int_{-2}^{-1} f(x)\,dx + \int_{-1}^{2} f(x)\,dx$ (M1)

$= 8.08$ (A1) N2

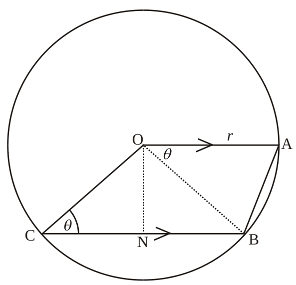
OR

area $= \int_{-2}^{2} |f(x)|\,dx$ (M1)

$= 8.08$ (A1) N2 2

[13]

43.



(a) $h = r\sin\theta \quad \text{CB} = 2\text{CN} = 2r\cos\theta$ (A1)(A1)

Using $T = (r + \text{CB})\dfrac{h}{2}$ (M1)

$T = \dfrac{r^2}{2}(\sin\theta + 2\sin\theta\cos\theta)$ (A1)

$= \dfrac{r^2}{2}(\sin\theta + \sin 2\theta)$ (AG) N0 4

(b) $\dfrac{dT}{d\theta} = \dfrac{r^2}{2}(\cos\theta + 2\cos 2\theta) = 0$ (for max) (M1)

$\Rightarrow \cos\theta + 2(2\cos^2\theta - 1) = 4\cos^2\theta + \cos\theta - 2 = 0$ (M1)(AG)

$\Rightarrow \cos\theta = 0.5931 \quad (\theta = 0.9359)$ (A1)

$\dfrac{d^2T}{d\theta^2} = \dfrac{r^2}{2}(-\sin\theta - 4\sin 2\theta)$ (M1)

$\theta = 0.9359 \quad \Rightarrow \quad \dfrac{d^2T}{d\theta^2} = -2.313r^2 < 0$

$\Rightarrow$ there is a **maximum** (when $\theta = 0.9359$) (R1) 5

(c) In triangle AOB: $\text{AB} = 2r\sin\dfrac{\theta}{2}$ (M1)(A1)

Perimeter OABC $= 2r + 2r\cos\theta + 2r\sin\dfrac{\theta}{2} = 75$ (M1)

When $\theta = 0.9359$, $r = 18.35$ cm (A1)

Area OABC $= \frac{r^2}{2}(\sin\theta + \sin 2\theta) = \frac{18.35^2}{2}(\sin 0.9359 + \sin 1.872)$ (M1)

$= 296\ \text{cm}^2$ (A1) N3 6

[15]

44. $f(x) = ax^3 + bx^2 + cx + d$ (M1)

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$ (M1)

$f(0) = 2 = d$ (A1)

$f'(1) = f(1) \rightarrow a + b + c + 2 = 3a + 2b + c$

$2 = 2a + b$

$f'(0) = -3 = c$ (A1)

$f''(-1) = 6 = -6a + 2b$

$b = \frac{12}{5}, a = -\frac{1}{5}$ (A1)(A1)

$f(x) = -\frac{1}{5}x^3 + \frac{12}{5}x^2 - 3x + 2 \left(\text{Accept } a = -\frac{1}{5}, b = \frac{12}{5}, c = -3, d = 2\right)$ (C6)

[6]

45. (a) r^{th} term $= \binom{n}{n-r} x^r h^{n-r} \left(= \frac{n!}{r!(n-r)!} x^r h^{n-r}\right)$ (A1)

(b) $\frac{\mathrm{d}(x^n)}{\mathrm{d}x} = \lim_{h\to 0}\left(\frac{(x+h)^n - x^n}{h}\right)$ (M1)

$= \lim_{h\to 0}\left(\frac{x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \ldots + h^n - x^n}{h}\right)$ (A1)

$= \lim_{h\to 0}\left(\frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + h^n - x^n}{h}\right)$ (A1)

$= \lim_{h\to 0}\left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \ldots + h^{n-1}\right)$ (A1)

Note: *Accept first, second and last terms in the 3 lines above.*

$= nx^{n-1}$ (A1)

(c) $x^n \times x^{-n} = 1$

$x^n \frac{\mathrm{d}(x^{-n})}{\mathrm{d}x} + x^{-n}\frac{\mathrm{d}(x^n)}{\mathrm{d}x} = 0$ (M1)

$x^n \frac{\mathrm{d}(x^{-n})}{\mathrm{d}x} + x^{-n} \times nx^{n-1} = 0$ (A1)

$x^n \frac{\mathrm{d}(x^{-n})}{\mathrm{d}x} + nx^{-1} = 0$ (A1)

$$\frac{\mathrm{d}\left(x^{-n}\right)}{\mathrm{d}x}=\frac{-nx^{-1}}{x^{n}}\left(=-nx^{-(1+n)}\right)$$ (A1)

[10]

46. (a) $f'(x) = 3\cos^2(4x + 1) \times (-\sin(4x + 1)) \times 4$ A1A1A1

$f'(x) = -12\cos^2(4x + 1)(\sin(4x + 1))$

Note: *Award A1 for 3 cos² (4x + 1), A1 for – sin (4x + 1) and A1 for 4.*

(b) $f'(x) = 0 \Rightarrow \cos^2(4x + 1) = 0$ **or** $\sin(4x + 1) = 0$

$\Rightarrow x=\frac{\pi}{8}-\frac{1}{4}, x=\frac{3\pi}{8}-\frac{1}{4}$ **or** $x=\frac{\pi-1}{4}$ A1A1A1

Note: *Do not penalize the inclusion of additional answers.*

[6]

47. $(\ln 3)\, 3^{x+y}\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 3x^2 + 3\frac{\mathrm{d}y}{\mathrm{d}x}$

(M1)A1A1A

1

Note: *Award A1 for (ln 3)*3^{x+y}*, A1 for* $\left(1+\frac{dy}{dx}\right)$ *and A1 for* $3x^2+3\frac{dy}{dx}$.

$\frac{\mathrm{d}y}{\mathrm{d}x}((\ln 3)\, 3^{x+y} - 3) = 3x^2 - (\ln 3)\, 3^{x+y}$ M1

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3x^2-(\ln 3)3^{x+y}}{(\ln 3)3^{x+y}-3}$$ A1 N0

[6]

48. (a) $f'(x)=\frac{1}{3x+1}\times 3\left(=\frac{3}{3x+1}\right)$ M1A1 N2

(b) Hence when $x = 2$, gradient of tangent $=\frac{3}{7}$ (A1)

$\Rightarrow$ gradient of normal is $-\frac{7}{3}$ (A1)

$y-\ln 7=-\frac{7}{3}(x-2)$ M1

$y=-\frac{7}{3}x+\frac{14}{3}+\ln 7$ A1 N4

(accept y = –2.33x + 6.61)

[6]

49. (a) $\frac{\mathrm{d}y}{\mathrm{d}x}=-\sin\theta+\mathrm{i}\cos\theta$ A1

EITHER

$\frac{\mathrm{d}y}{\mathrm{d}\theta}=-\mathrm{i}^2\sin\theta+\mathrm{i}\cos\theta$ A1

$= \mathrm{i}(\cos\theta + \mathrm{i}\sin\theta)$ A1

$= \mathrm{i}\, y$ AG N0

OR

$\mathrm{i}\,y = \mathrm{i}(\cos\theta + \mathrm{i}\sin\theta)$ $(= \mathrm{i}\cos\theta + \mathrm{i}^2\sin\theta)$ A1

$= \mathrm{i}\cos\theta - \sin\theta$ A1

$= \dfrac{\mathrm{d}y}{\mathrm{d}\theta}$ AG N0

(b) $\displaystyle\int \frac{\mathrm{d}y}{y} = \mathrm{i}\int \mathrm{d}\theta$ M1A1

$\ln y = \mathrm{i}\theta + c$ A1

Substituting (0, 1) $\quad 0 = 0 + c \Rightarrow c = 0$ A1

$\therefore \ln y = \mathrm{i}\theta$ A1

$y = \mathrm{e}^{\mathrm{i}\theta}$ AG N0

(c) $\cos n\theta + \mathrm{i}\sin n\theta = \mathrm{e}^{\mathrm{i}n\theta}$ M1

$= (\mathrm{e}^{\mathrm{i}\theta})^n$ A1

$= (\cos\theta + \mathrm{i}\sin\theta)^n$ AG N0

Note: *Accept this proof in reverse.*

(d) (i) $\cos 6\theta + \mathrm{i}\sin 6\theta = (\cos\theta + \mathrm{i}\sin\theta)^6$ M1

Expanding rhs using the binomial theorem M1A1

$= \cos^6\theta + 6\cos^5\theta\,\mathrm{i}\sin\theta + 15\cos^4\theta(\mathrm{i}\sin\theta)^2 + 20\cos^3\theta(\mathrm{i}\sin\theta)^3$

$+ 15\cos^2\theta(\mathrm{i}\sin\theta)^4 + 6\cos\theta(\mathrm{i}\sin\theta)^5 + (\mathrm{i}\sin\theta)^6$

Equating imaginary parts (M1)

$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$ A1

$\dfrac{\sin 6\theta}{\sin\theta} = 6\cos^5\theta - 20\cos^3\theta(1-\cos^2\theta) + 6\cos\theta(1-\cos^2\theta)^2$ A1

$= 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$ $(a = 32, b = -32, c = 6)$ A2 N0

(ii) $\displaystyle\lim_{\theta\to 0}\frac{\sin 6\theta}{\sin\theta} = \lim_{\theta\to 0}\left(32\cos^5\theta - 32\cos^3\theta + 6\cos\theta\right)$ M1

$= 32 - 32 + 6$

$= 6$ A1 N0

[20]

50. (a) $f'(x) = \ln x + x\left(\dfrac{1}{x}\right) - 1$ (M1)

$= \ln x$ A1 N2

(b) Using integration by parts

METHOD 1

$\displaystyle\int(\ln x)^2\,\mathrm{d}x = (\ln x)^2 \bullet x - \int x \bullet \frac{2}{x}(\ln x)\,\mathrm{d}x$ A1A1

$\displaystyle = x(\ln x)^2 - 2\int(\ln x)\,\mathrm{d}x$ (A1)

$= x(\ln x)^2 - 2(x\ln x - x) + C$ A1

$(= x(\ln x)^2 - 2x\ln x + 2x + C)$

METHOD 2

$\displaystyle\int(\ln x)^2\,\mathrm{d}x = x(\ln x)^2 - x\ln x - \int(\ln x - 1)\,\mathrm{d}x$ A1A1A1

$= x(\ln x)^2 - x\ln x - (x\ln x - x - x) + C$ A1

$(= x(\ln x)^2 - 2x\ln x + 2x + C)$

Note: *Do not penalize the absence of + C.*

[6]

51. $y = x\arcsin x$

$$\frac{dy}{dx} = \arcsin x + \frac{x}{\sqrt{1-x^2}}$$ M1A1

$$\frac{d}{dx}\left(\frac{x}{\sqrt{1-x^2}}\right) = \frac{1\left(\sqrt{1-x^2}\right) + x^2\left(1-x^2\right)^{\frac{1}{2}}}{1-x^2}$$ M1A1

$$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{1-x^2}} + \frac{1\left(\sqrt{1-x^2}\right) + x^2\left(1-x^2\right)^{\frac{1}{2}}}{1-x^2}$$

$$= \frac{1}{\left(1-x^2\right)^{\frac{1}{2}}} + \frac{1}{\left(1-x^2\right)^{\frac{1}{2}}} + \frac{x^2}{\left(1-x^2\right)^{\frac{3}{2}}}$$ A1

$$= \frac{2}{\left(1-x^2\right)^{\frac{1}{2}}} + \frac{x^2}{\left(1-x^2\right)^{\frac{3}{2}}}$$

$$= \frac{2\left(1-x^2\right) + x^2}{\left(1-x^2\right)^{\frac{3}{2}}}$$ A1

$$= \frac{2 - x^2}{\left(1-x^2\right)^{\frac{3}{2}}}$$ AG

Note: *The final A1A1 are for equivalent algebraic manipulations leading to AG.*

[6]

52. $e^{xy} - y^2 \ln x = 1$

Differentiating implicitly (M1)

$$e^{xy}\left(y + x\frac{dy}{dx}\right) - \left(2y\frac{dy}{dx}\ln x + \frac{y^2}{x}\right) = 0$$ A1A1

Notes: *Award A1 for* $e^{xy}\left(y + x\frac{dy}{dx}\right)$*, A1 for* $-\left(2y\frac{dy}{dx}\ln x + \frac{y^2}{x}\right)$ ***and*** *0.*

N.B. *Incorrect manipulation of* e^{xy} *can lead to the correct final result.*

EITHER

Collecting terms $e^{xy}y - \frac{y^2}{x} = \left(2y\ln x - xe^{xy}\right)\frac{dy}{dx}$ (M1)

$$\frac{dy}{dx} = \frac{ye^{xy} - \frac{y^2}{x}}{2y\ln x - xe^{xy}}$$ (A1)

$x = 1 \quad y = 1 \Rightarrow \frac{dy}{dx} = \frac{1-e}{e}$ A1

OR

Substituting $x = 1$ $y = 1$ (M1)

$e\left(1+\frac{dy}{dx}\right)-\left(2\frac{dy}{dx}\ln 1+1\right)=0$ (A1)

$\frac{dy}{dx}=\frac{1-e}{e}$ A1

[6]

53. (a) derivative = $\frac{3f'(x)[g(x)-1]-3f(x)g'(x)}{[g(x)-1]^2}$ M1

when $x = 0 \Rightarrow$ derivative = $\frac{3(1)(-4-1)-3(4)(5)}{(-4-1)^2}$ (A1)

$= -3$ A1 N0

(b) derivative = $f'(g(x) + 2x)(g'(x) + 2)$ M1

when $x = 1$ derivative = $f'(-1 + 2)(2 + 2)$ (A1)

$= (3)(4)$

$= 12$ A1 N0

[6]

54. (a) Use of quotient (or product) rule (M1)

$f'(x)=\frac{2(x^2+6)-(2x\times 2x)}{(x^2+6)^2}$ $\quad 2x(-1)(x^2+6)^{-2}(2x)+2(x^2+6)^{-1}$ A1

$=\frac{12-2x^2}{(x^2+6)^2}$ AG N0

(b) Solving $f'(x) = 0$ for x (M1)

$x=\pm\sqrt{6}$ A1

f has to be $1 - 1$ for f^{-1} to exist and so the least value of b is the larger of the two x-coordinates (accept a labelled sketch) R1

Hence $b=\sqrt{6}$ A1 N2

[6]

55. (a) Attempting implicit differentiation M1

$2x+y+x\frac{dy}{dx}+2y\frac{dy}{dx}=0$ A1

EITHER

Substituting $x = -1, y = k$ *eg* $-2+k-\frac{dy}{dx}+2k\frac{dy}{dx} = 0$ M1

Attempting to make $\frac{dy}{dx}$ the subject M1

OR

Attempting to make $\frac{dy}{dx}$ the subject *eg* $\frac{dy}{dx} = \frac{-(2x+y)}{x+2y}$ M1

Substituting $x = -1, y = k$ into $\frac{dy}{dx}$ M1

THEN

$\frac{dy}{dx}=\frac{2-k}{2k-1}$ A1 N1

(b) Solving $\frac{dy}{dx} = 0$ for k gives $k = 2$ A1

[6]

56. (a) AQ = $\sqrt{x^2+4}$ (km) (A1)

QY = (2 – x) (km) (A1)

$T = 5\sqrt{5}\,\text{AQ} + 5\text{QY}$ (M1)

$= 5\sqrt{5}\sqrt{(x^2+4)} + 5(2-x)$ (mins) A1

(b) Attempting to use the chain rule on $5\sqrt{5}\sqrt{(x^2+4)}$ (M1)

$\frac{\text{d}}{\text{d}x}\left(5\sqrt{5}\sqrt{(x^2+4)}\right) = 5\sqrt{5}\times\frac{1}{2}(x^2+4)^{-\frac{1}{2}}\times 2x$ A1

$\left(=\frac{5\sqrt{5}x}{\sqrt{x^2+4}}\right)$

$\frac{\text{d}}{\text{d}x}(5(2-x)) = -5$ A1

$\frac{\text{d}T}{\text{d}x} = \frac{5\sqrt{5}x}{\sqrt{x^2+4}} - 5$ AG N0

(c) (i) $\sqrt{5}\,x = \sqrt{x^2+4}$ or equivalent A1

Squaring both sides and rearranging to obtain $5x^2 = x^2 + 4$ M1

$x = 1$ A1 N1

Note: *Do not award the final A1 for stating a negative solution in final answer.*

(ii) $T = 5\sqrt{5}\sqrt{1+4} + 5(2-1)$ M1

= 30 (mins) A1 N1

Note: *Allow FT on incorrect x value.*

(iii) **METHOD 1**

Attempting to use the quotient rule M1

$u = x, v = \sqrt{x^2+4}, \frac{\text{d}u}{\text{d}x} = 1$ and $\frac{\text{d}v}{\text{d}x} = x(x^2+4)^{-1/2}$ (A1)

$\frac{\text{d}^2T}{\text{d}x^2} = 5\sqrt{5}\left[\frac{\sqrt{x^2+4} - \frac{1}{2}(x^2+4)^{-1/2}\times 2x^2}{(x^2+4)}\right]$ A1

Attempt to simplify (M1)

$= \frac{5\sqrt{5}}{(x^2+4)^{3/2}}\left[x^2+4-x^2\right]$ or equivalent A1

$= \frac{20\sqrt{5}}{(x^2+4)^{3/2}}$ AG

When $x = 1$, $\frac{20\sqrt{5}}{(x^2+4)^{3/2}} > 0$ and hence $T = 30$ is a minimum R1 N0

Note: *Allow FT on incorrect x value, $0 \le x \le 2$.*

METHOD 2

Attempting to use the product rule M1

$u=x, v=\sqrt{x^2+4}, \frac{du}{dx}=1$ and $\frac{dv}{dx}=x(x^2+4)^{-1/2}$ (A1)

$\frac{d^2T}{dx^2}=5\sqrt{5}(x^2+4)^{-1/2}-\frac{5\sqrt{5}x}{2}(x^2+4)^{-3/2}\times 2x$ A1

$\left(=\frac{5\sqrt{5}}{(x^2+4)^{1/2}}-\frac{5\sqrt{5}\,x^2}{(x^2+4)^{3/2}}\right)$

Attempt to simplify (M1)

$=\frac{5\sqrt{5}(x^2+4)-5\sqrt{5}\,x^2}{(x^2+4)^{3/2}}$ $\left(=\frac{5\sqrt{5}(x^2+4-x^2)}{(x^2+4)^{3/2}}\right)$ A1

$=\frac{20\sqrt{5}}{(x^2+4)^{3/2}}$ AG

When $x = 1$, $\frac{20\sqrt{5}}{(x^2+4)^{3/2}} > 0$ and hence $T = 30$ is a minimum R1 N0

Note: *Allow FT on incorrect x value, $0 \le x \le 2$.*

[18]

57. **METHOD 1**

$3x^2y^2+2x^3y\frac{dy}{dx}=-\pi\sin(\pi y)\frac{dy}{dx}$ A1A1A1

At (–1, 1), $3-2\frac{dy}{dx}=0$ M1A1

$\frac{dy}{dx}=\frac{3}{2}$ A1

METHOD 2

$3x^2y^2+2x^3y\frac{dy}{dx}=-\pi\sin(\pi y)\frac{dy}{dx}$ A1A1A1

$\frac{dy}{dx}=\frac{3x^2y^2}{-\pi\sin(\pi y)-2x^3y}$ A1

At (–1, 1), $\frac{dy}{dx}=\frac{3(-1)^2(1)^2}{-\pi\sin(\pi)-2(-1)^3(1)}=\frac{3}{2}$ M1A1

[12]