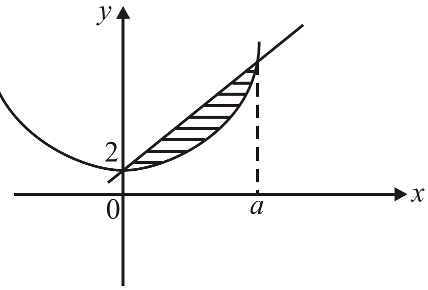
Integration Practice Problems (Legacy)

1. The area of the enclosed region shown in the diagram is defined by

$y \geq x^2 + 2$, $y \leq ax + 2$, where $a > 0$.



This region is rotated 360° about the x-axis to form a solid of revolution. Find, in terms of a, the volume of this solid of revolution.

Working:

Answer:
……………………………………………..

(Total 4 marks)

2. Using the substitution $u = \frac{1}{2}x + 1$, or otherwise, find the integral

$$\int x\sqrt{\frac{1}{2}x+1}\,\mathrm{d}x.$$

Working:

Answer:
……………………………………………..

(Total 4 marks)

3. When air is released from an inflated balloon it is found that the rate of decrease of the volume of the balloon is proportional to the volume of the balloon. This can be represented by the differential equation $\frac{\mathrm{d}v}{\mathrm{d}t} = -kv$, where v is the volume, t is the time and k is the constant of proportionality.

(a) If the initial volume of the balloon is v_0, find an expression, in terms of k, for the volume of the balloon at time t.

(b) Find an expression, in terms of k, for the time when the volume is $\frac{v_0}{2}$.

Working:

Answers:
(a) ……………………………………………..
(b) ……………………………………….........

(Total 4 marks)

4. Consider the function $f : x \mapsto x - x^2$ for $-1 \leq x \leq k$, where $1 < k \leq 3$.

(a) Sketch the graph of the function f.

(3)

(b) Find the total finite area enclosed by the graph of f, the x-axis and the line $x = k$.

(4)

(Total 7 marks)

5. The area between the graph of $y = e^x$ and the x-axis from $x = 0$ to $x = k$ ($k > 0$) is rotated through 360° about the x-axis. Find, in terms of k and e, the volume of the solid generated.

Working:

Answer:
....…………………………………………..........

(Total 4 marks)

6. Find the real number $k > 1$ for which $\int_1^k \left(1 + \frac{1}{x^2}\right) dx = \frac{3}{2}$.

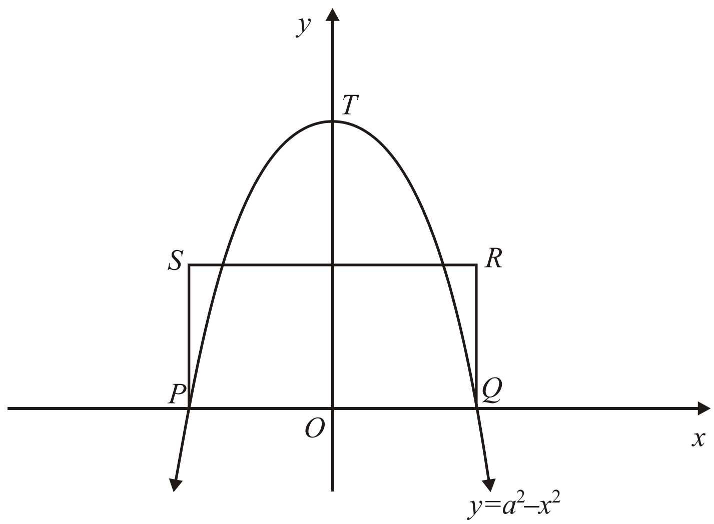
Working:

Answer:

....

(Total 4 marks)

7. In the diagram, *PTQ* is an arc of the parabola $y = a^2 - x^2$, where a is a positive constant, and *PQRS* is a rectangle. The area of the rectangle *PQRS* is equal to the area between the arc *PTQ* of the parabola and the x-axis.



Find, in terms of a, the dimensions of the rectangle.

Working:

Answer:

....

(Total 4 marks)

8. Consider the function $f_k(x) = \begin{cases} x \ln x - kx, & x > 0 \\ 0, & x = 0 \end{cases}$, where $k \in \mathbb{N}$

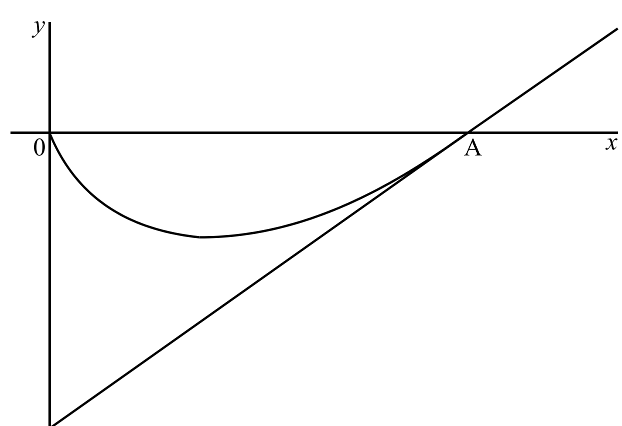
(a) Find the derivative of $f_k(x)$, $x > 0$.

(2)

(b) Find the interval over which $f_0(x)$ is increasing.

The graph of the function $f_k(x)$ is shown below.

(2)



(c) (i) Show that the stationary point of $f_k(x)$ is at $x = e^{k-1}$.

(ii) One x-intercept is at (0, 0). Find the coordinates of the other x-intercept.

(4)

(d) Find the area enclosed by the curve and the x-axis.

(5)

(e) Find the equation of the tangent to the curve at A.

(2)

(f) Show that the area of the triangular region created by the tangent and the coordinate axes is twice the area enclosed by the curve and the x-axis.

(2)

(g) Show that the x-intercepts of $f_k(x)$ for consecutive values of k form a geometric sequence.

(3)

(Total 20 marks)

9. Find the values of $a > 0$, such that $\int_a^{a^2} \frac{1}{1+x^2}\,dx = 0.22$.

Working:

Answer:

....

(Total 3 marks)

10. Let $f(x) = \ln|x^5 - 3x^2|$, $-0{\cdot}5 < x < 2$, $x \neq a$, $x \neq b$; (a, b are values of x for which $f(x)$ is not defined).

(a) (i) Sketch the graph of $f(x)$, indicating on your sketch the number of zeros of $f(x)$. Show also the position of any asymptotes.

(2)

(ii) Find all the zeros of $f(x)$, (that is, solve $f(x) = 0$).

(3)

(b) Find the **exact** values of a and b.

(3)

(c) Find $f'(x)$, and indicate clearly where $f'(x)$ is not defined.

(3)

(d) Find the **exact** value of the x-coordinate of the local maximum of $f(x)$, for $0 < x < 1.5$. (You may assume that there is no point of inflexion.)

(3)

(e) **Write down** the definite integral that represents the area of the region **enclosed** by $f(x)$ and the x-axis. (Do **not** evaluate the integral.)

(2)

(Total 16 marks)

11. Calculate the area bounded by the graph of $y = x\sin(x^2)$ and the x-axis, between $x = 0$ and the smallest positive x-intercept.

Working:

Answer:

....

(Total 3 marks)

12. Solve the differential equation $xy\frac{dy}{dx} = 1 + y^2$, given that $y = 0$ when $x = 2$.

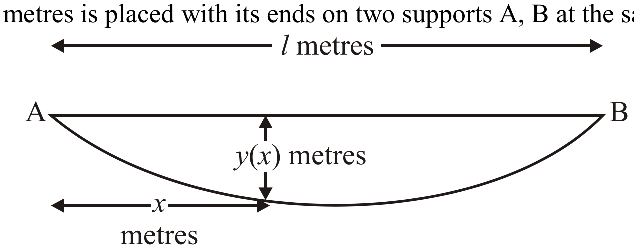
Working:

Answer:

....

(Total 3 marks)

13. A uniform rod of length l metres is placed with its ends on two supports A, B at the same horizontal level.



If $y(x)$ metres is the amount of sag (*ie* the distance below [AB]) at a distance x metres from support A, then it is known that

$$\frac{d^2y}{dx^2} = \frac{1}{125l^3}\left(x^2 - lx\right).$$

(a) (i) Let $z = \frac{1}{125l^3}\left(\frac{x^3}{3} - \frac{lx^2}{2}\right) + \frac{1}{1500}$. Show that $\frac{dz}{dx} = \frac{1}{125l^3}\left(x^2 - lx\right)$.

(ii) Given that $\frac{dw}{dx} = z$ and $w(0) = 0$, find $w(x)$.

(iii) Show that w satisfies $\frac{d^2w}{dx^2} = \frac{1}{125l^3}(x^2 - lx)$, and that $w(l) = w(0) = 0$.

(8)

(b) Find the sag at the centre of a rod of length 2.4 metres.

(2)

(Total 10 marks)

14. Find $\int \ln x \, dx$.

Working:

Answer: ……………………………………………..

(Total 3 marks)

15. The equation of motion of a particle with mass m, subjected to a force kx can be written as $kx = mv\frac{dv}{dx}$, where x is the displacement and v is the velocity. When $x = 0$, $v = v_0$. dx Find v, in terms of v_0, k and m, when $x = 2$.

Working:

Answer: ……………………………………………..

(Total 3 marks)

16. Find the value of a such that $\int_0^a \cos^2 x \, dx = 0.740$. Give your answer to 3 decimal places.

Working:

Answer: ……………………………………………..

(Total 3 marks)

17. Find the area of the region enclosed by the graphs of $y = \sin x$ and $y = x^2 - 2x + 1.5$, where $0 \le x \le \pi$.

Working:

Answer: ……………………………………………..

(Total 3 marks)

18. (a) Sketch and label the graphs of $f(x) = e^{-x^2}$ and $g(x) = e^{x^2} - 1$ for $0 \le x \le 1$, and shade the region A which is bounded by the graphs and the y-axis.

(3)

(b) Let the x-coordinate of the point of intersection of the curves $y = f(x)$ and $y = g(x)$ be p. Without finding the value of p, show that

$$\frac{p}{2} < \text{area of region } A < p.$$

(4)

(c) Find the value of p correct to four decimal places.

(2)

(d) Express the area of region A as a definite integral and calculate its value.

(3)

(Total 12 marks)

19. Let $f(t) = t^{\frac{1}{3}}\left(1 - \frac{1}{2t^{\frac{5}{3}}}\right)$. Find $\int f(t)\,\mathrm{d}t$.

Working:

Answer:

....

(Total 3 marks)

20. Let $f : x \mapsto \frac{\sin x}{x}$, $\pi \le x \le 3\pi$. Find the area enclosed by the graph of f and the x-axis.

Working:

Answer:

....

(Total 3 marks)

21. Find the general solution of the differential equation $\frac{\mathrm{d}x}{\mathrm{d}r} = kx(5 - x)$ where $0 < x < 5$, and k is a constant.

Working:

Answer:

....

(Total 3 marks)

22. Let $f(x) = x \cos 3x$.

(a) Use integration by parts to show that

$$\int f(x)\,\mathrm{d}x = \frac{1}{3}x \sin 3x + \frac{1}{9}\cos 3x + c.$$

(3)

(b) Use your answer to part (a) to calculate the **exact** area enclosed by $f(x)$ and the x-axis in each of the following cases. **Give your answers in terms of π.**

(i) $\frac{\pi}{6} \le x \le \frac{3\pi}{6}$

(ii) $\frac{3\pi}{6} \le x \le \frac{5\pi}{6}$

(iii) $\frac{5\pi}{6} \le x \le \frac{7\pi}{6}$.

(4)

(c) Given that the above areas are the first three terms of an arithmetic sequence, find an expression for the total area enclosed by $f(x)$ and the x-axis for $\frac{\pi}{6} \le x \le \frac{(2n+1)\pi}{6}$, where $n \in \mathbb{Z}^+$. **Give your answers in terms of n and π.**

(4)

(Total 11 marks)

23. A sample of radioactive material decays at a rate which is proportional to the amount of material present in the sample. Find the half-life of the material if 50 grams decay to 48 grams in 10 years.

Working:

Answer:

....

(Total 3 marks)

24. Find the area enclosed by the curves $y = \frac{2}{1+x^2}$ and $y = e^{\frac{x}{3}}$, given that $-3 \le x \le 3$.

Working:

Answer:

....

(Total 3 marks)

25. (a) Use integration by parts to find $\int x^2 \ln x \, dx$.

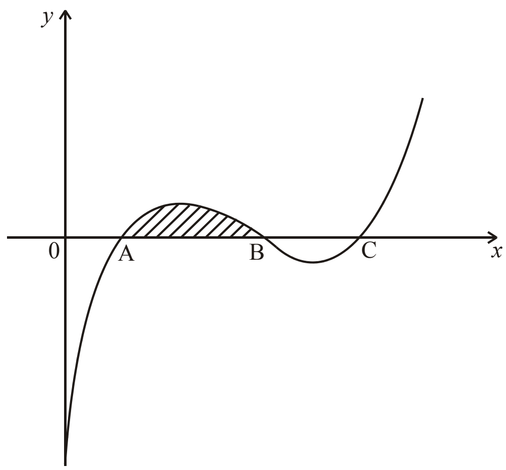
(b) Evaluate $\int_1^2 x^2 \ln x \, dx$

Working:

Answers:
(a) ....
(b) ....

(Total 6 marks)

26. The figure below shows part of the curve $y = x^3 - 7x^2 + 14x - 7$. The curve crosses the x-axis at the points A, B and C.



(a) Find the x-coordinate of A.
(b) Find the x-coordinate of B.
(c) Find the area of the shaded region.

Working:

Answers:
(a) ....
(b) ....
(c) ....

(Total 6 marks)

27. Find $\int (\theta \cos\theta - \theta) d\theta$.

Working:

Answer:
....

(Total 6 marks)

28. The tangent to the curve $y = f(x)$ at the point P(x, y) meets the x-axis at Q ($x - 1$, 0). The curve meets the y-axis at R(0, 2). Find the equation of the curve.

Working:

Answer:
....

(Total 6 marks)

29. (a) On the same axes sketch the graphs of the functions, $f(x)$ and $g(x)$, where

$f(x) = 4 - (1 - x)^2$, for $-2 \le x \le 4$,
$g(x) = \ln(x + 3) - 2$, for $-3 \le x \le 5$.

(2)

(b) (i) Write down the equation of any vertical asymptotes.
(ii) State the x-intercept and y-intercept of $g(x)$.

(3)

(c) Find the values of x for which $f(x) = g(x)$.

(2)

(d) Let A be the region where $f(x) \geq g(x)$ and $x \geq 0$.
(i) On your graph shade the region A.
(ii) Write down an integral that represents the area of A.
(iii) Evaluate this integral.

(4)

(e) In the region A find the maximum vertical distance between $f(x)$ and $g(x)$.

(3)
(Total 14 marks)

30. Using the substitution $y = 2 - x$, or otherwise, find $\int \left(\frac{x}{2-x}\right)^2 \mathrm{d}x$.

Working:

Answer:
....

(Total 6 marks)

31. The function f with domain $\left[0, \frac{\pi}{2}\right]$ is defined by $f(x) = \cos x + \sqrt{3}\sin x$.

This function may also be expressed in the form $R\cos(x - \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

(a) Find the **exact** value of R and of α.

(3)

(b) (i) Find the range of the function f.
(ii) State, giving a reason, whether or not the inverse function of f exists.

(5)

(c) Find the **exact** value of x satisfying the equation $f(x) = \sqrt{2}$.

(3)

(d) Using the result

$$\int \sec x \,\mathrm{d}x = \ln|\sec x + \tan x| + C, \text{ where } C \text{ is a constant,}$$

show that

$$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{f(x)} = \frac{1}{2}\ln(3 + 2\sqrt{3}).$$

(5)
(Total 16 marks)

32. Calculate the area enclosed by the curves $y = \ln x$ and $y = e^x - e$, $x > 0$.

Working:

Answer:
....

(Total 6 marks)

33. Given that $\frac{\mathrm{d}y}{\mathrm{d}x} = e^x - 2x$ and $y = 3$ when $x = 0$, find an expression for y in terms of x.

Working:

Answer:
....

(Total 6 marks)

34. (a) Find $\int_0^m \frac{\mathrm{d}x}{2x+3}$, giving your answer in terms of m.

(b) Given that $\int_0^m \frac{\mathrm{d}x}{2x+3} = 1$, calculate the value of m.

Working:

Answers:
(a) ....
(b) ....

(Total 6 marks)

35. Find $\int \frac{\ln x}{\sqrt{x}}\,dx$.

Working:

Answer:
....

(Total 6 marks)

36. The temperature T °C of an object in a room, after t minutes, satisfies the differential equation

$$\frac{dT}{dt} = k(T - 22), \text{ where } k \text{ is a constant.}$$

(a) Solve this equation to show that $T = Ae^{kt} + 22$, where A is a constant.

(3)

(b) When $t = 0$, $T = 100$, and when $t = 15$, $T = 70$.

(i) Use this information to find the value of A and of k.

(ii) Hence find the value of t when $T = 40$.

(7)

(Total 10 marks)

37. Find the total area of the two regions enclosed by the curve $y = x^3 - 3x^2 - 9x + 27$ and the line $y = x + 3$.

Working:

Answer:
....

(Total 6 marks)

38. Using the substitution $2x = \sin\theta$, or otherwise, find $\int\left(\sqrt{1-4x^2}\right)dx$.

Working:

Answer:
....

(Total 6 marks)

39. Consider the complex number $z = \cos\theta + i\sin\theta$.

(a) Using De Moivre's theorem show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta.$$

(2)

(b) By expanding $\left(z + \frac{1}{z}\right)^4$ show that

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3).$$

(4)

(c) Let $g(a) = \int_0^a \cos^4\theta\,d\theta$.

(i) Find $g(a)$.

(ii) Solve $g(a) = 1$

(5)

(Total 11 marks)

40. Consider the differential equation $\frac{dy}{d\theta} = \frac{y}{e^{2\theta} + 1}$.

(a) Use the substitution $x = e^{\theta}$ to show that

$$\int \frac{dy}{y} = \int \frac{dx}{x(x^2+1)}.$$

(3)

(b) Find $\int \frac{dx}{x(x^2+1)}$.

(4)

(c) Hence find y in terms of θ, if $y = \sqrt{2}$ when $\theta = 0$.

(4)
(Total 11 marks)

41. The function f' is given by $f'(x) = 2\sin\left(5x - \frac{\pi}{2}\right)$.

(a) Write down $f''(x)$.

(b) Given that $f\left(\frac{\pi}{2}\right) = 1$, find $f(x)$.

....
....
....
....
....
....
....
....
....
....
....
....

(Total 6 marks)

42. Use the substitution $u = x + 2$ to find $\int \frac{x^3}{(x+2)^2}\,dx$.

....
....
....
....
....
....
....
....
....
....
....
....

(Total 6 marks)

43. Solve the differential equation $x\frac{dy}{dx} - y^2 = 1$, given that $y = 0$ when $x = 2$. Give your answer in the form $y = f(x)$.

....
....
....
....
....
....
....
....
....
....
....
....

(Total 6 marks)

44. (a) Express as partial fractions $\dfrac{2x+4}{(x^2+4)(x-2)}$.

(b) Hence or otherwise, find $\displaystyle\int \frac{2x+4}{(x^2+4)(x-2)}\,\mathrm{d}x$.

Working:

Answers:

(a) ....

(b) ....

(Total 6 marks)

45. The function f is defined by $f(x) = \mathrm{e}^{px}(x+1)$, here $p \in \mathbb{R}$.

(a) (i) Show that $f'(x) = \mathrm{e}^{px}(p(x+1)+1)$.

(ii) Let $f^{(n)}(x)$ denote the result of differentiating $f(x)$ with respect to x, n times. Use mathematical induction to prove that

$$f^{(n)}(x) = p^{n-1}\mathrm{e}^{px}(p(x+1)+n),\ n \in \mathbb{Z}^+.$$

(7)

(b) When $p = \sqrt{3}$, there is a minimum point and a point of inflexion on the graph of f. Find the **exact** value of the x-coordinate of

(i) the minimum point;

(ii) the point of inflexion.

(4)

(c) Let $p = \dfrac{1}{2}$. Let R be the region enclosed by the curve, the x-axis and the lines $x = –2$ and $x = 2$. Find the area of R.

(2)

(Total 13 marks)

46. Find $\displaystyle\int \mathrm{e}^x \cos x\,\mathrm{d}x$.

Working:

Answer:

....

(Total 6 marks)

47. (a) Given that $\dfrac{x^2}{(1+x)(1+x^2)} \equiv \dfrac{a}{(1+x)} + \dfrac{bx+c}{(1+x^2)}$, calculate the value of a, of b and of c.

(5)

(b) (i) Hence, find $I = \displaystyle\int \frac{x^2}{(1+x)(1+x^2)}\,\mathrm{d}x$.

(ii) If $I = \dfrac{\pi}{4}$ when $x = 1$, calculate the value of the constant of integration giving your answer in the form $p + q \ln r$ where $p, q, r \in \mathbb{R}$

(7)
(Total 12 marks)

48. Let $f(x) = 2^{0.5x}$ and $g(x) = 3^{-0.5x} + \dfrac{5}{3}$. Let R be the region completely enclosed by the graphs of f and g, and the y-axis. Find the area of R.

(Total 6 marks)

49. Find $\displaystyle\int e^{2x} \sin x \,\mathrm{d}x$.

(Total 6 marks)

50. Solve the differential equation

$$(x+2)^2 \frac{\mathrm{d}y}{\mathrm{d}x} = 4xy \qquad (x > -2)$$

given that $y = 1$ when $x = -1$.

....

(Total 6 marks)

51. The region enclosed by the curves $y^2 = kx$ and $x^2 = ky$, where $k > 0$, is denoted by R. Given that the area of R is 12, find the value of k.

....
....
....
....
....
....
....
....
....
....
....
....
....

(Total 6 marks)

52. The function f is defined by $f(x) = \dfrac{\ln x}{x^3}$, $x \geq 1$.

(a) Find $f'(x)$ and $f''(x)$, simplifying your answers.

(6)

(b) (i) Find the **exact** value of the x-coordinate of the maximum point and justify that this is a maximum.

(ii) Solve $f''(x) = 0$, and show that at this value of x, there is a point of inflexion on the graph of f.

(iii) Sketch the graph of f, indicating the maximum point and the point of inflexion.

(11)

The region enclosed by the x-axis, the graph of f and the line $x = 3$ is denoted by R.

(c) Find the volume of the solid of revolution obtained when R is rotated through 360° about the x-axis.

(3)

(d) Show that the area of R is $\dfrac{1}{18}(4 - \ln 3)$.

(6)

(Total 26 marks)

53. Let $y = \cos\theta + \mathrm{i}\sin\theta$.

(a) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}\theta} = \mathrm{i}y$.

[You may assume that for the purposes of differentiation and integration, i may be treated in the same way as a real constant.]

(3)

(b) **Hence** show, using integration, that $y = \mathrm{e}^{\mathrm{i}\theta}$.

(5)

(c) Use this result to deduce de Moivre's theorem.

(2)

(d) (i) Given that $\dfrac{\sin 6\theta}{\sin\theta} = a\cos^5\theta + b\cos^3\theta + c\cos\theta$, where $\sin\theta \neq 0$, use de Moivre's theorem with $n = 6$ to find the values of the constants a, b and c.

(ii) Hence deduce the value of $\lim\limits_{\theta\to 0} \dfrac{\sin 6\theta}{\sin\theta}$.

(10)

(Total 20 marks)

54. Let $f(x) = x\ln x - x$, $x > 0$.

(a) Find $f'(x)$.

(b) Using integration by parts find $\int (\ln x)^2 \, dx$.

(Total 6 marks)

55. The function f is defined as $f(x) = \sin x \ln x$ for $x \in [0.5, 3.5]$.

(a) Write down the x-intercepts.

(b) The area above the x-axis is A and the **total** area below the x-axis is B. If $A = kB$, find k.

(Total 6 marks)

56. Solve the differential equation $(x^2 + 1)\dfrac{dy}{dx} - xy = 0$ where $x > 0$, $y > 0$, given that $y = 1$ when $x = 1$.

(Total 6 marks)

57. Solve the differential equation $\dfrac{dy}{dx} = 2xy^2$ given that $y = 1$ when $x = 0$.

Give your answer in the form $y = f(x)$.

(Total 6 marks)

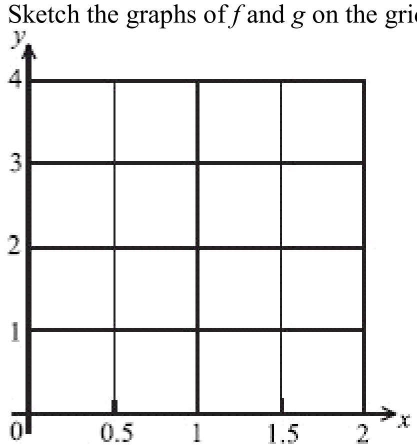
58. The graph of $y = \sin(3x)$ for $0 \le x \le \frac{\pi}{4}$ is is rotated through 2π radians about the x-axis.

Find the **exact** volume of the solid of revolution formed.

(Total 6 marks)

59. For $x \ge \frac{1}{2}$, let $f(x) = x^2 \ln(x+1)$ and $g(x) = \sqrt{2x-1}$.

(a) Sketch the graphs of f and g on the grid below.



(b) Let A be the region completely enclosed by the graphs of f and g.
Find the area of A.

(Total 6 marks)

60. Find $\int_0^{\ln 3} \frac{e^x}{e^{2x}+9}\,dx$, expressing your answer in **exact** form.

....

....

....

....

....

....

....

....

....

....

....

....

....

(Total 6 marks)

61. (a) Using the formula for cos $(A + B)$ prove that $\cos^2\theta = \frac{\cos 2\theta + 1}{2}$.

(3)

(b) Hence, find $\int \cos^2 x\,dx$.

(4)

Let $f(x) = \cos x$ and $g(x) = \sec x$ for $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

Let R be the region enclosed by the two functions.

(c) Find the **exact** values of the x-coordinates of the points of intersection.

(4)

(d) Sketch the functions f and g and clearly shade the region R.

(3)

The region R is rotated through 2π about the x-axis to generate a solid.

(e) (i) Write down an integral which represents the volume of this solid.

(ii) Hence find the **exact** value of the volume.

(10)

(Total 24 marks)

62. (a) Use integration by parts to show that

$$\int \sin x \cos x\, e^{-\sin x}\,dx = -e^{-\sin x}(1 + \sin x) + C.$$

(4)

Consider the differential equation $\frac{dy}{dx} - y\cos x = \sin x \cos x$.

(b) Find an integrating factor.

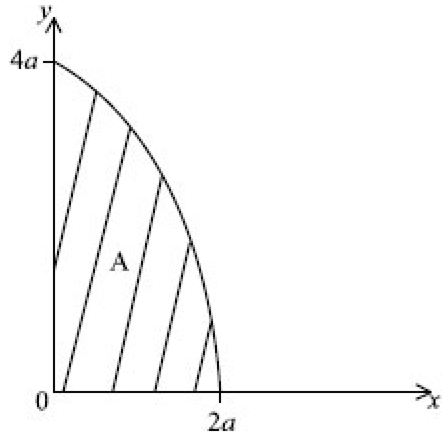
(3)

(c) Solve the differential equation, given that $y = -2$ when $x = 0$. Give your answer in the form $y = f(x)$.

(9)

(Total 16 marks)

63. The diagram below shows the shaded region A which is bounded by the axes and part of the curve $y^2 = 8a(2a - x)$, $a > 0$. Find in terms of a the volume of the solid formed when A is rotated through 360° around the x-axis.



(Total 6 marks)

64. Find $\int_0^a \arcsin x\,\mathrm{d}x,\ 0 < a < 1$.

(Total 6 marks)

65. Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+y^2}{1+x^2}$, given that $y = \sqrt{3}$ when $x = \frac{\sqrt{3}}{3}$.

Give your answer in the form $y = \frac{ax+\sqrt{a}}{a-x\sqrt{a}}$ where $a \in \mathbb{Z}^+$.

....

....

(Total 6 marks)

66. Find the area between the curves $y = 2 + x - x^2$ and $y = 2 - 3x + x^2$.

....

....

....

....

....

....

....

....

....

....

....

....

(Total 7 marks)

67. The region bounded by the curve $y = \frac{ln(x)}{x}$ and the lines $x = 1$, $x = e$, $y = 0$ is rotated through 2π radians about the x-axis.
Find the volume of the solid generated.

....

....

....

....

....

....

....

....

....

....

....

....

(Total 12 marks)

68. Show that $\int_0^{\frac{\pi}{6}} x \sin 2x \, dx = \frac{\sqrt{3}}{8} - \frac{\pi}{24}$.

....

....

....

....

....

....

....

....

....

....

....

....

(Total 6 marks)

69. By using an appropriate substitution find

$$\int \frac{\tan(\ln y)}{y} dy, y > 0.$$

....

....

....

....

(Total 6 marks)

70. The curve $y = \mathrm{e}^{-x} - x + 1$ intersects the x-axis at P.

(a) Find the x-coordinate of P.

(2)

(b) Find the area of the region completely enclosed by the curve and the coordinate axes.

(3)

(Total 5 marks)

71. A particle moves in a straight line in a positive direction from a fixed point O.

The velocity v m s^{-1}, at time t seconds, where $t \geq 0$, satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{-v\left(1+v^2\right)}{50}.$$

The particle starts from O with an initial velocity of 10 m s^{-1}.

(a) (i) Express as a definite integral, the time taken for the particle's velocity to decrease from 10 m s^{-1} to 5 m s^{-1}.

(ii) **Hence** calculate the time taken for the particle's velocity to decrease from 10 m s^{-1} to 5 m s^{-1}.

(5)

(b) (i) Show that, when $v > 0$, the motion of this particle can also be described by the differential equation $\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{-\left(1+v^2\right)}{50}$ where x metres is the displacement from O.

(ii) Given that $v = 10$ when $x = 0$, solve the differential equation expressing x in terms of v.

(iii) **Hence** show that $v = \dfrac{10 - \tan\frac{x}{50}}{1 + 10\tan\frac{x}{50}}$.

(14)

(Total 19 marks)

72. (a) Using l'Hopital's Rule, show that $\lim\limits_{x\to\infty} x\mathrm{e}^{-x} = 0$.

(2)

(b) Determine $\int_0^a x\mathrm{e}^{-x}\,\mathrm{d}x$.

(5)

(c) Show that the integral $\int_0^\infty x\mathrm{e}^{-x}\,\mathrm{d}x$ is convergent and find its value.

(2)

(Total 9 marks)

Integration Practice Problems (Legacy) - MarkScheme

1. Let the volume of the solid of revolution be V.

$V = \pi \int_0^a \left((ax+2)^2 - (x^2+2)^2\right) \mathrm{d}x$ (M1)

$= \pi \int_0^a (a^2x^2 + 4ax + 4 - x^4 - 4x^2 - 4)\mathrm{d}x$ (M1)

$= \pi \left[\frac{1}{3}a^2x^3 + 2ax^2 - \frac{1}{5}x^5 - \frac{4}{3}x^3\right]_0^a$ (M1)

$= \pi\left(\frac{2}{15}a^5 + \frac{2}{3}a^3\right)$ units^3 (A1)

$= \frac{2a^3\pi}{15}(a^2 + 5)$ (C4)

Note: *The last line is not required*

[4]

2. Let $u = \frac{1}{2}x + 1 \Leftrightarrow x = 2(u-1) \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = 2$

Then $\int x\left(\frac{1}{2}x+1\right)^{1/2} \mathrm{d}x = \int 2(u-1) \times u^{1/2} \times 2\mathrm{d}u$ (M1)

$= 4\int (u^{3/2} - u^{1/2})\,\mathrm{d}u$ (A1)

$= 4\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + C$ (M1)

$= 4\left[\frac{2}{5}\left(\frac{1}{2}x+1\right)^{5/2} - \frac{2}{3}\left(\frac{1}{2}x+1\right)^{3/2}\right] + C$ (A1)

$= \frac{8}{15}\left(\frac{1}{2}x+1\right)^{3/2}\left(\frac{3}{2}x-2\right) + C$ (C4)

Note: *The last line is not required*

[4]

3. (a) Given $\frac{\mathrm{d}v}{\mathrm{d}t} = -kv$

$\Leftrightarrow \int \frac{\mathrm{d}v}{v} = -k\int \mathrm{d}t$

$\Leftrightarrow \ln v = -kt + C$ (M1)

$\Leftrightarrow v = Ae^{-kt} (A = e^C)$

At $t = 0$, $v = v_0 \Rightarrow A = v_0$

$\Leftrightarrow v = v_0e^{-kt}$ (A1) (C2)

(b) Put $v = \frac{v_0}{2}$

then $\frac{v_0}{2} = v_0e^{-kt}$ (M1)

$\Leftrightarrow \frac{1}{2} = e^{-kt}$

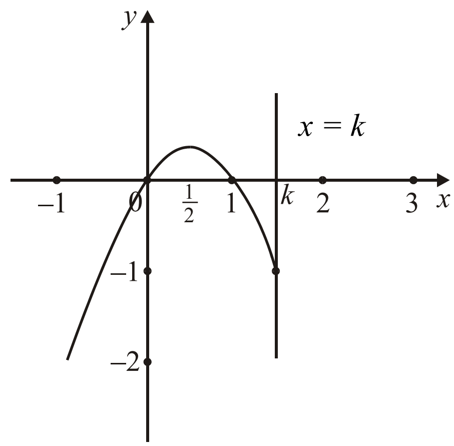
$\Leftrightarrow \ln\frac{1}{2} = -kt$

$\Leftrightarrow t = \dfrac{\ln 2}{k}$ (A1) (C2)

Note: *Accept equivalent forms, eg* $t = \dfrac{\ln \frac{1}{2}}{-k}$

[4]

4. (a)



3

Notes: *Award (A1) for the correct intercepts*
(A1) for graphing over the correct interval
(A1) for the correct x-coordinate of the maximum point.

(b) Required area $= \int_0^1 (x - x^2)\mathrm{d}x - \int_1^k (x - x^2)\mathrm{d}x$ (M1)

$= \left[\dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_0^1 - \left[\dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_1^k$ (A1)

$= \dfrac{1}{6} - \dfrac{k^2}{2} + \dfrac{k^3}{3} + \dfrac{1}{6} = \dfrac{1}{3} + \dfrac{k^3}{3} - \dfrac{k^2}{2}$ (M1)

$= \dfrac{1}{6}(2 + 2k^3 - 3k^2)$ (A1)

OR

Required Area $= \left|\int_1^k (x - x^2)\mathrm{d}x\right|$ (M1)

$= \left|\left[\dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_1^k\right|$ (M1)(A1)

$= \dfrac{k^3}{3} - \dfrac{k^2}{2} + \dfrac{1}{6}$ (A1) 4

[7]

5. $V = \pi \int_0^k \mathrm{e}^{2x}\mathrm{d}x$ (M1)

$= \dfrac{\pi}{2}[\mathrm{e}^{2x}]_0^k$ (A1)

$= \dfrac{\pi}{2}(\mathrm{e}^{2k} - 1)$ (M1)(A1) (C4)

[4]

6. $\int_1^k \left(1+\frac{1}{x^2}\right)\mathrm{d}x = \frac{3}{2}$

$\left[x - \frac{1}{x}\right]_1^k = \frac{3}{2}$ (M1)

$k - \frac{1}{k} = \frac{3}{2}$ (A1)

$2k^2 - 3k - 2 = 0$

$(2k + 1)(k - 2) = 0$ (M1)

$k = 2$ since $k > 1$ (A1) (C4)

[4]

7. Area under parabola $= 2\int_0^a (a^2 - x^2)\,\mathrm{d}x$ (M1)

$= 2\left[a^2x - \frac{1}{3}x^3\right]_0^a$ (A1)

$= \frac{4}{3}a^3$ (A1)

Since $PQ = 2a$, the dimensions of the rectangle are $2a \times \frac{2}{3}a^2$. (A1) (C4)

[4]

8. (a) $f_k(x) = x \ln x - kx$

$\Rightarrow f_k'(x) = \ln x + 1 - k$ (M1)(A1) 2

(b) $f_0'(x) = \ln x + 1$

$f_0(x)$ increases where $f_0'(x) > 0$ (M1)

$\Rightarrow \ln x > -1 \Rightarrow x > \frac{1}{\mathrm{e}}$ (A1) 2

(c) (i) Stationary points happen where $f_k'(x) = 0$

$\Rightarrow \ln x = k - 1$ (M1)

$\Rightarrow x = \mathrm{e}^{k-1}$ (A1)

(ii) x intercepts are where $f_k(x) = 0$

$\Rightarrow x \ln x - kx = 0$

$\Rightarrow x(\ln x - k) = 0$ (M1)

$\Rightarrow x = 0$ or $\ln x = k$

$\Rightarrow x = \mathrm{e}^k$

$\Rightarrow (\mathrm{e}^k, 0)$ (A1) 4

(d) Area $= \int_0^{\mathrm{e}^k} |x\ln x - kx|\mathrm{d}x = \left(= \int_0^{\mathrm{e}^k}(kx - x\ln x)\mathrm{d}x\right)$ (M1)

Integrate $x \ln x$ by parts.

$\int x\ln x\mathrm{d}x = \frac{x^2}{2}\ln x - \int \frac{x}{2}\mathrm{d}x$ (M1)

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$ (A1)

$\Rightarrow$ Area $= \left|\left[\frac{x^2}{2}\ln x - \frac{x^2}{4} - \frac{kx^2}{2}\right]_0^{\mathrm{e}^k}\right|$

$= \dfrac{e^{2k}}{4}$ (A1) 5

Note: *Given $x \ln x - kx = f_k(x) \approx 0$ when $x = 0$.*

(e) Gradient of the tangent at $A(e^k, 0)$, m is $f_k'(e^k) = \ln e^k + 1 - k$ (M1)

$= 1$

Therefore, an equation of the tangent is $y = x - e^k$. (A1) 2

(f) The tangent forms a right angle triangle with the coordinate axes.

The perpendicular sides are each of length e^k. (M1)

Area of the triangle $= \dfrac{1}{2} \times e^k \times e^k = \dfrac{1}{2} e^{2k}$ (A1)

$\dfrac{1}{2} e^{2k} = 2\left(\dfrac{1}{4} e^{2k}\right)$ *ie* The area of the triangle is twice the area enclosed by the curve and the x-axis. (AG) 2

(g) Since the x-intercepts are of the form $x_k = e^k$, for $k \in \mathbb{N}$ (M1)

then $x_{k+1} = e^{k+1}$

and $\dfrac{x_{k+1}}{x_k} = e$ (A1)

Therefore, the x-intercepts $x_0, x_1, \ldots x_k, \ldots$ form a geometric sequence with $x_0 = 1$ and a common ratio of e. (R1) 3

[20]

9. If $\int_a^{a^2} \dfrac{1}{1+x^2} dx = 0.22$

Then $[\arctan x]_a^{a^2} = 0.22$ (M1)

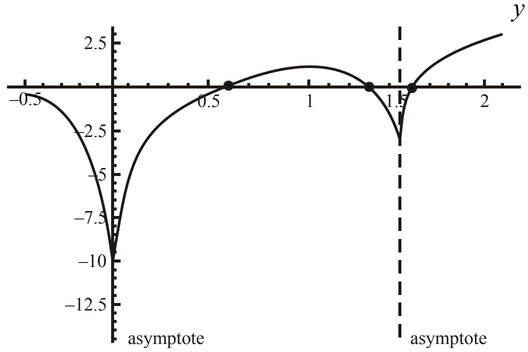
$\arctan a^2 - \arctan a - 0.22 = 0$ (A1)

$a = 2.04$ or $a = 2.62$ (A1) (C3)

Notes: *Award final (A1) only if both correct answers are shown.*
If no working is shown and only one answer is correct, award (C1).
GDC example: finding solutions from a graph.

[3]

10. (a) (i)



(G2)

Note: *Award (G1) for correct shape, including three zeros, and (G1) for both asymptotes*

(ii) $f(x) = 0$ for $x = 0.599, 1.35, 1.51$ (G1)(G1)(G1) 5

(b) $f(x)$ is undefined for

$(x^5 - 3x^2) = 0$ (M1)

$x^2(x^3 - 3) = 0$

Therefore, $x = 0$ or $x = 3^{1/3}$ (A2) 3

(c) $f'(x) = \dfrac{5x^4 - 6x}{x^5 - 3x^2}\left(\text{or}\ \dfrac{5x^3 - 6}{x^4 - 3x}\right)$ (M1)(A1)

$f'(x)$ is undefined at $x = 0$ and $x = 3^{1/3}$ (A1) 3

(d) For the x-coordinate of the local maximum of $f(x)$, where $0 < x < 1.5$ put $f'(x) = 0$ (R1)

$5x^3 - 6 = 0$ (M1)

$x = \left(\dfrac{6}{5}\right)^{\frac{1}{3}}$ (A1) 3

(e) The required area is

$A = \displaystyle\int_{0.599}^{1.35} f(x)\mathrm{d}x$ (A2) 2

Note: *Award (A1) for each correct limit.*

[16]

11. $x \sin(x^2) = 0$ when $x^2 = 0$ $(+k\pi, k \in \mathbb{Z})$, *ie* $x = 0\ (+\sqrt{k\pi})$ (A1)

The required area = $\displaystyle\int_0^{\sqrt{\pi}} x \sin(x^2)\,\mathrm{d}x$ (M1)

$= 1$ (G1) (C3)

OR

Area = $\displaystyle\int_0^{\sqrt{\pi}} x \sin(x^2)\,\mathrm{d}x$

$= -\dfrac{1}{2}\left[\cos(x^2)\right]_0^{\sqrt{\pi}}$ (M1)

$= -\dfrac{1}{2}\ (-1 - 1)$

$= 1$ (A1) (C3)

[3]

12. $xy\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2 \Rightarrow \displaystyle\int \frac{y}{1+y^2}\mathrm{d}y = \int \frac{1}{x}\mathrm{d}x$ (M1)

$\dfrac{1}{2}\ln(1 + y^2) = \ln x + \ln c$ (M1)

$1 + y^2 = kx^2\ (k = c^2)$

$y = 0$ when $x = 2$, and so $1 = 4k$

Thus, $1 + y^2 = \dfrac{1}{4}x^2$ or $x^2 - 4y^2 = 4$. (A1) (C3)

[3]

13. (a) (i) $\dfrac{\mathrm{d}z}{\mathrm{d}x} = \dfrac{1}{125l^3}(x^2 - lx)$ (M1)(AG)

(ii) $w(x) = \displaystyle\int z(x)\mathrm{d}x + C = \frac{1}{125l^3}\left(\frac{x^4}{12} - \frac{lx^3}{6}\right) + \frac{x}{1500} + C$ (M1)(A1)

Hence, $C = w(0) = 0$ (A1)

and therefore, $w(x) = \dfrac{1}{125l^3}\left(\dfrac{x^4}{12} - \dfrac{lx^3}{6}\right) + \dfrac{x}{1500}$ (A1)

(iii) $\dfrac{\mathrm{d}^2w}{\mathrm{d}x^2} = \dfrac{\mathrm{d}z}{\mathrm{d}x} = \dfrac{1}{125l^3}(x^2 - lx)$ (A1)

We have seen above that $w(0) = 0$

$$w(l) = \frac{1}{125l^3}\left(\frac{l^4}{12} - \frac{l^4}{6}\right) + \frac{l}{1500} = -\frac{l}{1500} + \frac{l}{1500} = 0$$ (A2) 8

(b) When $l = 2.4$, $x = 1.2$ at the centre of the rod.

Now, $y(1.2) = \frac{1}{125(2.4)^3}\left(\frac{1.2^4}{12} - \frac{2.4(1.2)^3}{6}\right) + \frac{1.2}{1500}$ (M1)

$= 0.0005$ m. (A1) 2

[10]

14. Let $\int \ln x\mathrm{d}x = \int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x$ where $u = \ln x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = 1$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$ and $v = x$. (M1)

Using integration by parts,

$\int \ln x\mathrm{d}x = x\ln x - \int x\frac{1}{x}\,\mathrm{d}x$ (A1)

$= x\ln x - x + C$ (A1)

[3]

15. If $kx = mv\frac{\mathrm{d}v}{\mathrm{d}x}$

Then $\int kx\,\mathrm{d}x = \int mv\,\mathrm{d}v$ (using separation of variables) (M1)

$\Rightarrow \frac{1}{2}kx^2 = \frac{1}{2}mv^2 + C$ (A1)

When $x = 0$, $v = v_0$, therefore $C = -\frac{1}{2}mv_0^2$

Therefore $v^2 = v_0^2 + \frac{kx^2}{m}$

Therefore when $x = 2$, $v = \sqrt{v_0^2 + \frac{4k}{m}}$ (A1)

[3]

16. $\int_0^a \cos^2 x\,\mathrm{d}x = 0.740$

$\Rightarrow \frac{1}{2}\int_0^a (\cos 2x + 1)\,\mathrm{d}x = 0.740$ (using formulae and statistical tables)

$\Rightarrow \frac{1}{2}\left[\frac{1}{2}\sin 2x + x\right]_0^a = 0.740$ (A1)

$\Rightarrow \sin(2a) + 2a - 2.960 = 0$ (A1)

$\Rightarrow a = 1.047$ (using a graphic display calculator) (A1)

[3]

17. Solving $\sin x = x^2 - 2x + 1.5$

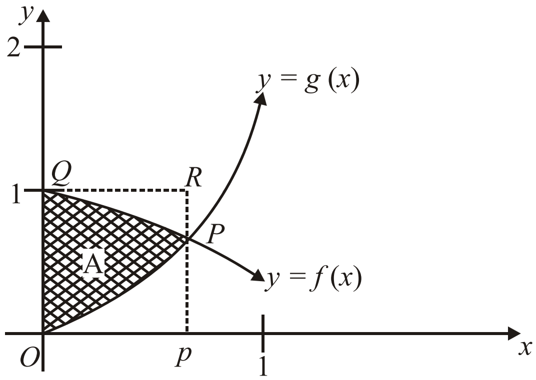
gives $x = 0.6617$ or 1.7010 (using a graphic display calculator) (A1)

Then $A = \int_{0.6617}^{1.7010}(\sin x - (x^2 - 2x + 1.5))\,\mathrm{d}x$ (M1)

$= 0.271$ units^2 (using a graphic display calculator) (A1)

[3]

18. (a)



(A3)

***Notes:** Award (A1) for y = f (x), with (0, 1) shown.*
Award (A1) for y = g (x), and (A1) for region A

(b) area ΔOPQ < area of region A < area of rectangle *OSRQ* (M1)(R1)

$\frac{1}{2}(1)(p)$ < area of region $A < (p)(1)$ (A2)

$\frac{p}{2}$ < area of region $A < p$ (AG)

(c) Solving the equation $e^{-p^2} - e^{p^2} + 1 = 0$ using a calculator gives
$p = 0.6937$ (4 decimal places) (A2)
OR the value of p may be found as follows:

$e^{-p^2} = e^{p^2} - 1$

$\Rightarrow e^{2p^2} - e^{p^2} - 1 = 0$

$\Rightarrow e^{p^2} = \frac{1 \pm \sqrt{1+4}}{2}$

$\Rightarrow e^{p^2} = \frac{1 \pm \sqrt{5}}{2}$ since $e^{p^2} > 0$

$\Rightarrow$ This gives $p = \sqrt{\ln\left(\frac{1+\sqrt{5}}{2}\right)} \approx 0.6937$ (4 decimal places) (A2)

(d) Area of region $A = \int_0^p \left(e^{-x^2} - \left[e^{x^2} - 1\right]\right)dx$ (M2)

$= 0.467$ (using a graphic display calculator) (A1)

[12]

19. $\int t^{\frac{1}{3}}\left(1 - \frac{1}{2t^{\frac{5}{3}}}\right)dt = \int t^{\frac{1}{3}}\left(1 - \frac{t^{-\frac{5}{3}}}{2}\right)dt$

$= \int \left(t^{\frac{1}{3}} - \frac{t^{-\frac{4}{3}}}{2}\right)dt$ (M1)

$= \frac{3}{4}t^{\frac{4}{3}} + \frac{3}{2}t^{-\frac{1}{3}} + C$ (M1)(A1) (C3)

***Note**: Do not penalize for the absence of +C.*

[3]

20. x-intercepts are $= \pi, 2\pi, 3\pi$. (A1)

Area required = $\left|\int_{\pi}^{2\pi} \frac{\sin x}{x}\,dx\right| + \int_{2\pi}^{3\pi} \frac{\sin x}{x}\,dx$ (M1)

= 0.4338 + 0.2566

= 0.690 units^2 (G1) (C3)

[3]

21. Given $\frac{dx}{dt} = kx(5 - x)$

then $\frac{1}{x(5-x)}\frac{dx}{dt} = k$ (M1)

$\int \frac{1}{5x} + \frac{1}{5(5-x)}\,dx = \int k\,dt$ (A1)

$\frac{1}{5}\ln x - \frac{1}{5}\ln(5-x) = kt + C$ **or** $\left(\frac{x}{5-x}\right)^{\frac{1}{5}} = Ae^{kt}$ **or** $\left(\frac{x}{5-x}\right) = Ae^{5kt}$ (A1) (C3)

[3]

22. (a) Using integration by parts

$\int x\cos 3x\,dx = \frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3x\,dx$ (M1)(A2)

$= \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C$ (not required) (AG) 3

(b) (i) Area = $\left|\left[\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\right]_{\frac{\pi}{6}}^{\frac{3\pi}{6}}\right| = \frac{2\pi}{9}$ (M1)(A1)

(ii) Area = $\left|\left[\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\right]_{\frac{3\pi}{6}}^{\frac{5\pi}{6}}\right| = \frac{4\pi}{9}$ (A1)

(iii) Area = $\left|\left[\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\right]_{\frac{5\pi}{6}}^{\frac{7\pi}{6}}\right| = \frac{6\pi}{9}$ (A1) 4

Note: *Accept negative answers for part (b), as long as they are exact.*
Do ***not*** *accept answers found using a calculator.*

(c) The above areas form an arithmetic sequence with

$u_1 = \frac{2\pi}{9}$ and $d = \frac{2\pi}{9}$ (A1)

The required area = $S_n = \frac{n}{2}\left[\frac{4\pi}{9} + \frac{2\pi}{9}(n-1)\right]$ (M1)(A1)

$= \frac{n\pi}{9}(n+1)$ (A1) 4

[11]

23. If A g is present at any time, then $\frac{dA}{dt} = kA$ where k is a constant.

Then, $\int \frac{dA}{A} = k\int dt$

$\Rightarrow \ln A = kt + c$

$\Rightarrow A = e^{kt+c} = c_1 e^{kt}$

When $t = 0$, $c_1 = 50 \Rightarrow 48 = 50e^{10k}$. (A1)

$\frac{\ln 0.96}{10} = k$ or $k = -0.00408(2)$ (A1)

For half life, $25 = 50e^{kt}$

$\Rightarrow \ln 0.5 = kt$

$\Rightarrow t = \frac{10 \ln 0.5}{\ln 0.96} = 169.8$.

Therefore, half-life = 170 years (3 sf) (A1) (C3)

[3]

24. The curves meet when $x = -1.5247$ and $x = 0.74757$. (G1)

The required area = $\int_{-1.5247}^{0.74757} \left(\frac{2}{1+x^2} - e^{\frac{x}{3}} \right) dx$ (M1)

= 1.22. (G1) (C3)

[3]

25. (a) $\int x^2 \ln x \, dx = \left[\frac{x^3}{3} \ln x \right] - \int \frac{x^3}{3} \frac{1}{x} dx$ (M1)(A1)(A1)

$= \frac{x^3}{3} \ln x - \frac{x^3}{9}$ (Constant of integration not required.) (A1) (C4)

(b) $\int_1^2 x^2 \ln x \, dx = 1.07 \quad \left(\text{or } \frac{8}{3} \ln 2 - \frac{7}{9} \right)$ (A2) (C2)

[6]

26. (a) At A, $x = 0.753$ (G2) (C2)

(b) At B, $x = 2.45$ (G2) (C2)

(c) Area $\int_{0.753}^{2.45} y dx = 1.78$ (M1)(G1) (C2)

[6]

27. Using integration by parts $u = \theta$ $\quad v = \sin \theta$ (M1)

$du = d\theta \quad dv = \cos \theta d\theta$

$\Rightarrow \int \theta \cos \theta d\theta = \theta \sin \theta - \int \sin \theta d\theta$ (M1)(A1)

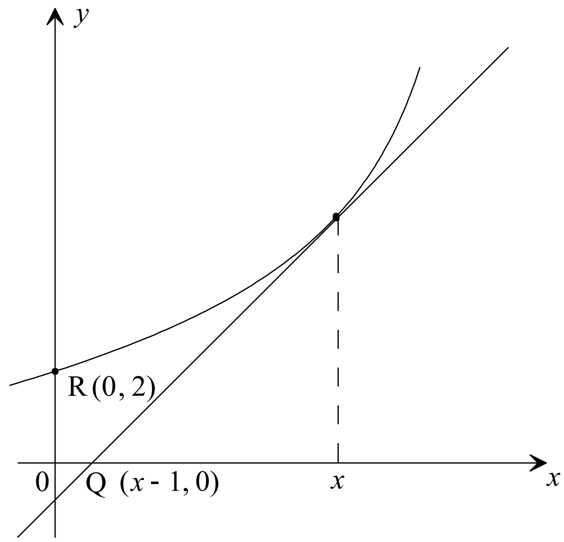
$\Rightarrow \int \theta \cos \theta d\theta = \theta \sin \theta + \cos \theta$ (A1)

Therefore, $\Rightarrow \int (\theta \cos \theta - \theta) d\theta = \theta \sin \theta + \cos \theta - \frac{\theta^2}{2} + c$ (A2) (C6)

Note: *Award (C5) for* $\theta \sin \theta + \cos \theta - \frac{\theta^2}{2}$*, ie penalize omission of + c by [1 mark].*

[6]

28.



From the diagram,

$$\frac{dy}{dx} = \frac{y}{1}$$ (M1)(A1)

=> $\int \frac{dy}{y} = \int dx$ (M1)

=> $\ln y = x + c$ (A1)

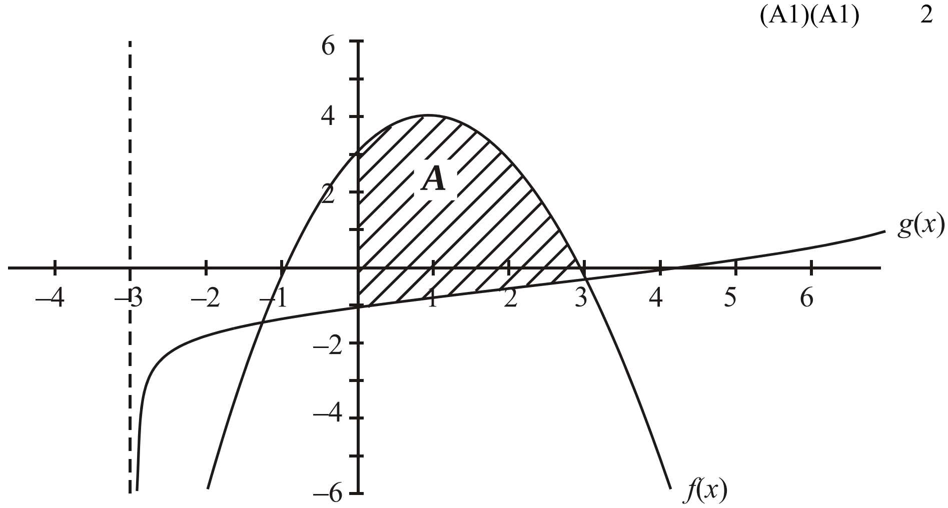
=> $y = e^{x+c} = Ae^x$. (A1)

But R (0, 2) lies on the curve and so $A = 2$. (A1)

Thus $y = 2e^x$ (C6)

[6]

29. (a) (A1)(A1) 2



Note: *Award (A1) for showing the basic shape of f (x).*
Award (A1) for showing both the vertical asymptote ***and*** *the basic shape of g (x).*

(b) (i) $x = -3$ is the vertical asymptote. (A1)

(ii) x-intercept: $x = 4.39$ ($= e^2 - 3$) (G1)

y-intercept: $y = -0.901$ ($= \ln 3 - 2$) (G1) 3

(c) $f(x) = g(x)$

$x = -1.34$ or $x = 3.05$ (G1)(G1) 2

(d) (i) See graph

(ii) Area of $A = \int_0^{3.05} \left(4 - (1 - x)^2\right) - (\ln (x + 3) - 2)dx$ (M1)(A1)

(iii) Area of $A = 10.6$ (G1) 4

(e) $y = f(x) - g(x)$

$y = 5 + 2x - x^2 - \ln(x + 3)$

$\frac{dy}{dx} = 2 - 2x - \frac{1}{x+3}$ (M1)

Maximum occurs when $\frac{dy}{dx} = 0$

$2 - 2x = \frac{1}{x+3}$

$5 - 4x - 2x^2 = 0$

$x = 0.871$ (A1)

$y = 4.63$ (A1)

OR

Vertical distance is the difference $f(x) - g(x)$. (M1)

Maximum of $f(x) - g(x)$ occurs at $x = 0.871$. (G1)

The maximum value is 4.63. (G1) 3

[14]

30. $I = \int\left(\frac{2-y}{y}\right)^2 (-dy)$ (M1)(A1)

$= -\int\left(\frac{4}{y^2} - \frac{4}{y} + 1\right) dy$

$= \frac{4}{y} + 4\ln|y| - y + c$ (A1)(A1)(A1)

$= \frac{4}{2-x} + 4\ln|2-x| - (2-x) + c$ (A1) (C6)

Note: c and modulus signs not required.

[6]

31. (a) $\cos x + \sqrt{3}\sin x = R\cos\alpha\cos x + R\sin\alpha\sin x$ (M1)

$\Rightarrow R\cos\alpha = 1, R\sin\alpha = \sqrt{3}$

$\Rightarrow R = 2, \alpha = \frac{\pi}{3}$ (A1)(A1) 3

Note: *Award (M1)(A1)(A0) if degrees used instead of radians.*

(b) (i) Since $f(x) = 2\cos\left(x - \frac{\pi}{3}\right)$,

$f_{max} = 2\left(\text{when } x = \frac{\pi}{3}\right); f_{min} = 1$ (when $x = 0$) (A1)(A1)

Range is [1, 2] (A1)

(ii) Inverse does not exist because f is not 1:1 (R2) 5

Notes: *Award (R2) for a correct answer with a valid reason.*
Award (R1) for a correct answer with an attempt at a valid reason, eg horizontal line test.
Award (R0) for just saying inverse does not exist, without any reason.

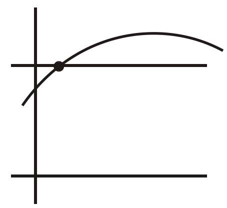
(c) $f(x) = \sqrt{2} \Rightarrow \cos\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{2}}{2}$ (M1)

$x - \frac{\pi}{3} = \pm\frac{\pi}{4}$ (A1)

$x = \frac{\pi}{12}$ (A1)

OR

$f(x) = \sqrt{2}$



(M1)

$x = 0.262$ (G1)

$\Rightarrow x = \frac{\pi}{12}$ (A1) 3

(d) $I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sec\left(x - \frac{\pi}{3}\right)dx$ (M1)

$= \frac{1}{2}\left[\ln\left(\sec\left(x - \frac{\pi}{3}\right) + \tan\left(x - \frac{\pi}{3}\right)\right)\right]_0^{\frac{\pi}{2}}$ (A1)

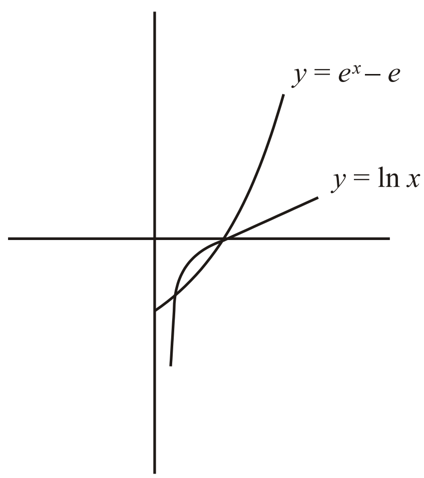
$= \frac{1}{2}\ln\left(\frac{\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{2 - \sqrt{3}}\right)$ (A1)(A1)

$= \frac{1}{2}\ln\left(\frac{\sqrt{3}(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}\right) = \frac{1}{2}\ln(3 + 2\sqrt{3})$. (M1)(AG) 5

Note: *Award zero marks for any work using GDC.*

[16]

32.



Curves intersect at $x = 0.233$ (G1)

and $x = 1$ (G1)

Area $= \int_{0.233}^{1}(\ln x - e^x + e)dx$ (M1)(A1)

$= 0.201$ (G2) (C6)

[6]

33. $y = e^x - x^2 + C$ (A1)(A1)(A1)

$3 = e^0 - 0 + C$ (M1)

$C = 2$ (A1)

$y = e^x - x^2 + 2$ (A1) (C6)

[6]

34. (a) $\int_0^m \frac{dx}{2x+3} = \left[\frac{1}{2}\ln|2x+3|\right]_0^m$ (M1)(A1)

$= \frac{1}{2}\ln\left|\frac{2m+3}{3}\right| \left(\text{or } \frac{1}{2}\ln|2m+3| - \frac{1}{2}\ln 3\right)$ (A1) (C3)

Note: *Modulus signs are not required.*

(b) $\frac{1}{2}\ln\left|\frac{2m+3}{3}\right| = 1$

$\left|\frac{2m+3}{3}\right| = e^2, \left(\frac{2m+3}{3} = \pm e^2, \text{negative sign not required}\right)$ (M1)

$\frac{2m+3}{3} = e^2 \Rightarrow \frac{2m}{3} = e^2 - 1$ (A1)

$m = \frac{3}{2}(e^2 - 1)\ (= 9.58)$ (A1) (C3)

Note: *Do not penalize if a candidate has also obtained the incorrect value m = –12.6.*

[6]

35. $\int \frac{\ln x}{\sqrt{x}}dx = \int u\frac{dv}{dx}dx = \int \ln x \, d\frac{(2x^{\frac{1}{2}})}{dx}dx$ (M1)

$= 2x^{\frac{1}{2}}\ln x - 2\int x^{\frac{1}{2}} \times \frac{1}{x}dx$ (A1)(A1)

$= 2x^{\frac{1}{2}}\ln x - 2\int \frac{1}{\sqrt{x}}dx$ (A1)

$= 2x^{\frac{1}{2}}\ln x - 4x^{\frac{1}{2}} + C$ (A2) (C6)

Note: *Award only (A1) if the C is missing.*

[6]

36. (a) $\frac{dT}{dt} = k(T-22) \Rightarrow \int \frac{dT}{T-22} = \int k\,dt$ (M1)

$\ln|T - 22| = kt + c$ (accept $\ln(T-22)$) (A1)

$T - 22 = Ae^{kt}$ (A1)

$T = Ae^{kt} + 22$ (AG) 3

(b) (i) When $t = 0$, $T = 100 \Rightarrow 100 = Ae^0 + 22$ (M1)

$A = 78$ (A1)

When $t = 15$, $T = 70 \Rightarrow 70 = 78e^{15k} + 22$ (M1)

$\Rightarrow k = \frac{1}{15}\ln\frac{48}{78}$ (= –0.0324) (A1)

(ii) $40 = 78e^{-0.0324} + 22$ (A1)

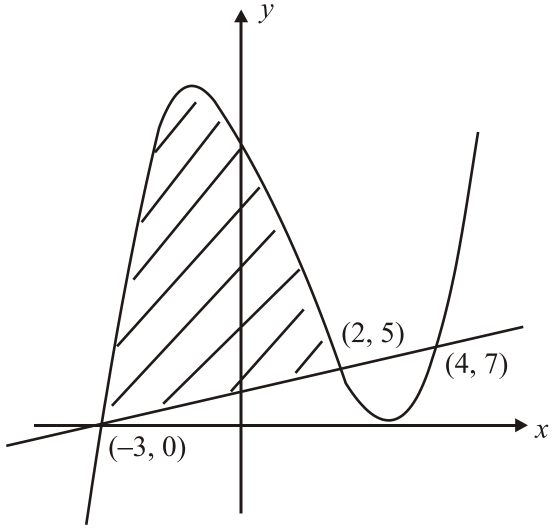
$e^{-0.0324} = \frac{18}{78}$ (A1)

$t = -\frac{1}{0.0324}\ln\frac{18}{78}(= 45.3)$ (A1) 7

[10]

37. METHOD 1

Region required is given by



from gdc outer intersections are at $x = -3$ and $x = 4$ (A1)(A1)

$\Rightarrow \text{Area} = \int_{-3}^{4} |y_1 - y_2|\,dx$ (M1)

$= 101.75$ (A3) (C6)

METHOD 2

From gdc intersections are at $x = -3$, $x = 2$, $x = 4$ (A1)(A1)(A1)

$\text{Area} = \int_{3}^{2}\left(x^3 - 3x^2 - 9x + 27 - (x+3)\right)dx + \int_{2}^{4}\left(x + 3 - (x^3 - 3x^2 - 9x + 27)\right)dx$ (M1)(M1)

$= 101.75$ (A1) (C6)

[6]

38. $\int\left(\sqrt{1-4x^2}\right)dx$

Let $2x = \sin\theta$

$\Rightarrow 2\frac{dx}{d\theta} = \cos\theta \quad \Rightarrow dx = \frac{1}{2}\cos\theta\,d\theta$

$\Rightarrow \int\left(\sqrt{1-4x^2}\right)dx = \int\sqrt{1-\sin^2\theta}\,\frac{1}{2}\cos\theta\,d\theta$

$= \int\frac{1}{2}\cos^2\theta\,d\theta$ (A1)

$= \int\frac{1}{4}(\cos 2\theta + 1)\,d\theta$ (A1)

$= \frac{1}{8}\sin 2\theta + \frac{\theta}{4} + C$ (A1)(A1)

$= \frac{1}{4}\left[2x\sqrt{1-4x^2} + \arcsin 2x\right] + C$ (A1)(A1) (C6)

[6]

39. (a) $z^n = \cos n\theta + i\sin n\theta$

$\frac{1}{z^n} = \cos(-n\theta) + i\sin(-n\theta)$ (M1)

$= \cos n\theta - \mathrm{i}\sin(n\theta)$ (A1)

Therefore $z^n + \frac{1}{2^n} = 2\cos n\theta$ (AG) 2

(b) $\left(z+\frac{1}{z}\right)^4 = z^4 + 4z^3\left(\frac{1}{z}\right) + 6(z^2)\left(\frac{1}{z^2}\right) + 4z\left(\frac{1}{z^3}\right) + \frac{1}{z^4}$ (M1)

$= z^4 + \frac{1}{z^4} + 4\left(z^2 + \frac{1}{z^2}\right) + 6$ (M1)

$(2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$ (A1)

$\cos^4\theta = \frac{1}{16}(2\cos 4\theta + 8\cos 2\theta + 6)$ (A1)

$= \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$ (AG) 4

(c) (i) $\int_0^a \cos^4\theta \,\mathrm{d}\theta = \frac{1}{8}\int_0^a (\cos 4\theta + 4\cos 2\theta + 3)\,\mathrm{d}\theta$ (M1)

$= \frac{1}{8}\left[\frac{1}{4}\sin 4\theta + 2\sin 2\theta + 3\theta\right]_0^a$ (A1)

$g(a) = \frac{1}{8}\left(\frac{1}{4}\sin 4a + 2\sin 2a + 3a\right)$ (A1)

(ii) $1 = \frac{1}{8}\left(\frac{1}{4}\sin 4a + 2\sin 2a + 3a\right)$

$a = 2.96$ (A1)

Since $\cos^4\theta \geq 0$ then $g(a)$ is an increasing function so there is only one root. (R1) 5

[11]

40. (a) $(\mathrm{e}^{2\theta} + 1)\mathrm{d}y = y\mathrm{d}\theta$

Separating variables yields $\int \frac{\mathrm{d}y}{y} = \int \frac{\mathrm{d}\theta}{\mathrm{e}^{2\theta}+1}$ (M1)

$x = \mathrm{e}^{\theta} \Rightarrow \mathrm{e}^{2\theta} + 1 = x^2 + 1$ (A1)

$\frac{\mathrm{d}x}{\mathrm{d}\theta} = \mathrm{e}^{\theta}$ (A1)

$\mathrm{d}\theta = \frac{\mathrm{d}x}{x}$

$\int \frac{\mathrm{d}y}{y} = \int \frac{\mathrm{d}x}{x(x^2+1)}$ (AG) 3

(b) Using partial fractions let

$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$ (M1)

$A(x^2+1) + Bx^2 + Cx = 1$

$A = 1,\ B = -1,\ C = 0$ (A1)

$\int \frac{1}{x(x^2+1)}\mathrm{d}x = \int\left(\frac{1}{x} - \frac{x}{x^2+1}\right)\mathrm{d}x$

$= \ln x - \frac{1}{2}\ln(x^2+1) + C$ (A1)(A1) 4

(c) Therefore $\ln y = \ln x - \frac{1}{2}\ln(x^2+1) + \ln k$

$\ln y = \ln\left(\frac{kx}{\sqrt{x^2+1}}\right)$ (A1)

$y = \frac{kx}{\sqrt{x^2+1}}$

When $\theta = 0$, $x = 1$, $y = \sqrt{2} \Rightarrow \sqrt{2} = \frac{k}{\sqrt{2}} \Rightarrow k = 2.$ (M1)

Therefore $y = \frac{2e^{\theta}}{\sqrt{e^{2\theta}+1}}$ (A1) 4

[11]

41. (a) Using the chain rule $f'(x) = \left(2\cos\left(5x - \frac{\pi}{2}\right)\right)5$ (M1)

$= 10\cos\left(5x - \frac{\pi}{2}\right)$ A1 2

(b) $f(x) = \int f'(x)\,dx$

$= -\frac{2}{5}\cos\left(5x - \frac{\pi}{2}\right) + c$ A1

Substituting to find c, $f\left(\frac{\pi}{2}\right) = -\frac{2}{5}\cos\left(5\left(\frac{\pi}{2}\right) - \frac{\pi}{2}\right) + c = 1$ M1

$c = 1 + \frac{2}{5}\cos 2\pi = 1 + \frac{2}{5} = \frac{7}{5}$ (A1)

$f(x) = -\frac{2}{5}\cos\left(5x - \frac{\pi}{2}\right) + \frac{7}{5}$ A1 N2 4

[6]

42. Substituting $u = x + 2 \Rightarrow u - 2 = x$, $du = dx$ (M1)

$\int \frac{x^3}{(x+2)^2}\,dx = \int \frac{(u-2)^3}{u^2}\,du$ A1

$= \int \frac{u^3 - 6u^2 + 12u - 8}{u^2}\,du$ A1

$= \int u\,du + \int(-6)\,du + \int \frac{12}{u}\,du - \int 8u^{-2}\,du$ A1

$= \frac{u^2}{2} - 6u + 12\ln|u| + 8u^{-1} + c$ A1

$= \frac{(x+2)^2}{2} - 6(x+2) + 12\ln|x+2| + \frac{8}{x+2} + c$ A1 N0

[6]

43. $x\frac{dy}{dx} - y^2 = 1, \Rightarrow x\frac{dy}{dx} = y^2 + 1$

Separating variables (M1)

$\frac{dy}{y^2+1} = \frac{dx}{x}$ A1

$\arctan y = \ln x + c$ A1A1

$y = 0, x = 2 \Rightarrow \arctan 0 = \ln 2 + c$

$-\ln 2 = c$ (A1)

$\arctan y = \ln x - \ln 2 = \ln \frac{x}{2}$

$y = \tan\left(\ln \frac{x}{2}\right)$ A1 N3

[6]

44. (a) Let $\frac{2x+4}{(x^2+4)(x-2)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-2}$ (M1)

$\Rightarrow 2x+4 = (Ax+B)(x-2) + C(x^2+4)$

comparing coefficients

$A + C = 0 \quad -2A + B = 2$ and $-2B + 4C = 4$

$\Rightarrow A = -1 \quad B = 0$ and $C = 1 \quad \left(-\frac{x}{x^2+4} + \frac{1}{x-2}\right)$ (A1)(A1)(A1)(C4)

(b) $\int \frac{2x+4}{(x^2+4)(x-2)}\mathrm{d}x = \int -\frac{x}{x^2+4}\mathrm{d}x + \int \frac{\mathrm{d}x}{x-2}$

$= -\frac{1}{2}\ln\left|x^2+4\right| + \ln\left|x-2\right| + C \quad \left(= \ln \frac{A(x-2)}{\sqrt{x^2+4}}\right)$ (A1)(A1) (C2)

[6]

45. (a) (i) $f'(x) = p\mathrm{e}^{px}(x+1) + \mathrm{e}^{px}$ (A1)

$= \mathrm{e}^{px}\left(p(x+1)+1\right)$ (AG)

(ii) The result is true for $n = 1$ since

$\text{LHS} = \mathrm{e}^{px}\left(p(x+1)+1\right)$

and $\text{RHS} = p^{1-1}\mathrm{e}^{px}\left(p(x+1)+1\right) = \mathrm{e}^{px}\left(p(x+1)+1\right)$. (M1)

Assume true for $n = k: f^{(k)}(x) = p^{k-1}\mathrm{e}^{px}\left(p(x+1)+k\right)$ (M1)

$f^{(k+1)}(x) = \left(f^{(k)}(x)\right)' = p^{k-1}p\mathrm{e}^{px}\left(p(x+1)+k\right) + p^{k-1}\mathrm{e}^{px}p$ (M1)(A1)

$= p^k\mathrm{e}^{px}\left(p(x+1)+k+1\right)$ (A1)

Therefore, true for $n = k \Rightarrow$ true for $n = k+1$ and the proposition is proved by induction. (R1) 7

(b) (i) $f'(x) = \mathrm{e}^{\sqrt{3}x}\left(\sqrt{3}(x+1)+1\right) = 0$ (M1)

$\Rightarrow x = -\frac{1+\sqrt{3}}{\sqrt{3}} \quad \left(= -\frac{\sqrt{3}+3}{3}\right)$ (A1) N1

(ii) $f''(x) = \sqrt{3}\mathrm{e}^{\sqrt{3}x}\left(\sqrt{3}(x+1)+2\right) = 0$ (M1)

$\Rightarrow x = -\frac{2+\sqrt{3}}{\sqrt{3}} \quad \left(= -\frac{2\sqrt{3}+3}{3}\right)$ (A1) N1 4

(c) $f(x) = \mathrm{e}^{0.5x}(x+1)$

EITHER

area $= -\int_{-2}^{-1} f(x)\mathrm{d}x + \int_{-1}^{2} f(x)\mathrm{d}x$ (M1)

$= 8.08$ (A1) N2

OR

area $=\int_{-2}^{2}|f(x)|\,\mathrm{d}x$ (M1)

$=8.08$ (A1) N2 2

[13]

46. $\int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx$ (M1)(A1)

$= e^x \cos x + e^x \sin x - \int e^x \cos x\,dx$ (M1)(A1)

$2\int e^x \cos x\,dx = e^x(\cos x + \sin x) + c$ (M1)

$\int e^x \cos x\,dx = \frac{e^x}{2}(\cos x + \sin x) + k$ (A1) (C6)

Note: *Do not penalize for missing integration constants.*

[6]

47. (a) $\frac{x^2}{(1+x)(1+x^2)} \equiv \frac{a}{(1+x)} + \frac{bx+c}{(1+x^2)}$

$x^2 \equiv a(1+x^2) + (bx+c)(1+x)$ (M1)(A1)

$1 = a + b,\ 0 = a + c,\ 0 = b + c$

Solving gives $1 = 2a$

$a = \frac{1}{2} \Rightarrow b = \frac{1}{2}, c = -\frac{1}{2}.$ (A1)(A1)(A1) (N2)

(b) (i) $I = \frac{1}{2}\int \frac{1}{(1+x)} + \frac{x-1}{(1+x^2)}\,dx$

$= \frac{1}{2}\int \frac{1}{(1+x)}\,dx + \frac{1}{4}\int \frac{2x}{(1+x^2)}\,dx - \frac{1}{2}\int \frac{dx}{(1+x^2)}$ (M1)

$= \frac{1}{2}\ln|1+x| + \frac{1}{4}\ln|1+x^2| - \frac{1}{2}\arctan x + k$ (A1)(A1)(A1)

Note: *Do not penalize the absence of k, or the absolute value signs.*

(ii) $\frac{\pi}{4} = \frac{1}{2}\ln 2 + \frac{1}{4}\ln 2 - \frac{\pi}{8} + k$ (M1)(A1)

$\frac{3\pi}{8} = \frac{3}{4}\ln 2 + k$

$\frac{3\pi}{8} - \frac{3}{4}\ln 2 = k \quad \left(accept\ p = \frac{3\pi}{8}, q = -\frac{3}{4}, r = 2\right)$ (A1) (N1)

Note: *I is not unique. Accept equivalent expressions which may lead to different values of p, q, r.*

[12]

48. Attempting to find point of intersection (M1)

Intersection at $x = 2$ (A1)

Note: *Award M1A1 if $x = 2$ is seen as upper limit of an integral.*

Using appropriate definite integrals M2

Area = 1.66 A2 N2

[6]

49. **METHOD 1**

$\int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx$ (M1)A1

$= -e^{2x}\cos x + 2\left(e^{2x}\sin x - 2\int e^{2x}\sin x\,dx\right)$ (M1)A1

$= -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x\,dx$

$5\int e^{2x}\sin x\,dx = e^{2x}(2\sin x - \cos x)$ (M1)

$\int e^{2x}\sin x\,dx = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$ A1 N0

METHOD 2

$\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int e^{2x}\cos x\,dx$ (M1)A1

$= \frac{1}{2}e^{2x}\sin x - \frac{1}{4}\int e^{2x}\cos x - \frac{1}{4}\int e^{2x}\sin x\,dx$ (M1)A1

$\Rightarrow \frac{5}{4}\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x$ (M1)

$\int e^{2x}\sin x\,dx = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$ A1

Note: *Do not penalize the absence of constants of integration.*

[6]

50. $\int \frac{1}{y}\,dy = \int \frac{4x}{(x+2)^2}\,dx$ M1

Let $u = x + 2 \Rightarrow x = u - 2$

$du = dx$ M1

$\ln y = \int \frac{4(u-2)}{u^2}\,du$

$= \int \frac{4}{u} - \frac{8}{u^2}\,du$ A1

$\ln y = 4\ln u + 8u^{-1} + c$ A1

$\ln y = 4\ln(x+2) + \frac{8}{x+2} + c$ A1

$(-1, 1) \Rightarrow c = -8$ A1

$\Rightarrow \ln y = 4\ln(x+2) + \frac{8}{x+2} - 8$ N0

[6]

51. Curves meet at (0, 0) and (k, k) (A1)

Area $= \int_0^k \left(k^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^2}{k}\right)dx = 12$ M1A1

$\Rightarrow \left[k^{\frac{1}{2}}\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3k}\right]_0^k = 12$ A1

$\Rightarrow \left[\frac{2}{3}k^2 - \frac{k^2}{3}\right] = 12$ A1

$\Rightarrow \frac{k^2}{3} = 12$

$\Rightarrow k = 6$ A1 N0

[6]

52. (a) Using quotient rule

$$f'(x) = \frac{x^3 \times \frac{1}{x} - 3x^2 \ln x}{x^6}$$ A1

$$= \frac{1 - 3\ln x}{x^4}$$ A1 N2

$$f''(x) = \frac{-\frac{3}{x} \times x^4 - 4x^3(1 - 3\ln x)}{x^8}$$ M1A1

$$= \frac{-7 + 12\ln x}{x^5}$$ A1 N2

(b) (i) For a maximum, $f'(x) = 0$ giving (M1)

$$\ln x = \frac{1}{3}$$

$$x = e^{\frac{1}{3}}$$ A1 N2

EITHER

$$f''\left(e^{\frac{1}{3}}\right) = \frac{12 \times \frac{1}{3} - 7}{e^{\frac{5}{3}}} < 0$$ M1A1

$\therefore$ maximum AG N0

OR

for $x < e^{\frac{1}{3}}$, $f'(x) > 0$

for $x > e^{\frac{1}{3}}$, $f'(x) < 0$ M1A1

$\therefore$ maximum AG N0

(ii) $f''(0) = 0 \Rightarrow \ln(x) = \frac{7}{12}$ M1

$x = e^{\frac{7}{12}}\ (1.79)$ A1

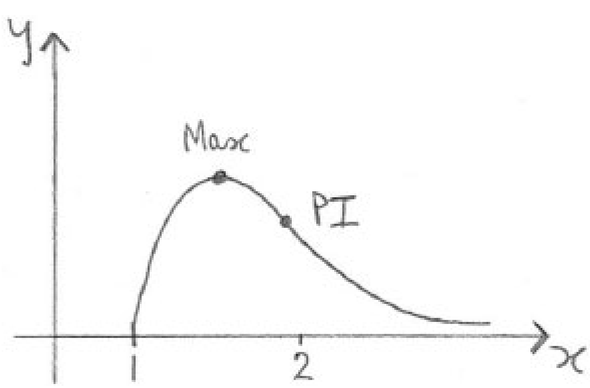
$f''(1.5) = -0.281$ A1

$f''(2) = 0.0412$ A1

Note: *Accept any two sensible values either side of 1.79.*

$\therefore$ Change of sign $\Rightarrow$ point of inflexion R1

(iii)



A1A1

Note: *Award A1 for shape, A1 for marking values which locate the maximum point and point of inflexion correctly.*

(c) Using $V = \int_1^3 \pi y^2\,dx$ (M1)

$= \int_1^3 \pi\left(\frac{\ln x}{x^3}\right)^2 dx$ (A1)

$= 0.0458$ A1 N2

(d) Area $A = \int_1^3 \frac{\ln x}{x^3}\,dx$ A1

Using integrating by parts (M1)

$A = \left[-\frac{\ln x}{2x^2}\right]_1^3 + \frac{1}{2}\int_1^3 \frac{1}{x^3}\,dx$ A1

$= -\frac{\ln 3}{18} - \frac{1}{4}\left[\frac{1}{x^2}\right]_1^3$ A1A1

$= -\frac{\ln 3}{18} - \frac{1}{4}\left(\frac{1}{9} - 1\right) \quad \left(= -\frac{\ln 3}{18} + \frac{2}{9}\right)$ A1

$= \frac{1}{18}(4 - \ln 3)$ AG N0

[26]

53. (a) $\frac{dy}{dx} = -\sin\theta + i\cos\theta$ A1

EITHER

$\frac{dy}{d\theta} = -i^2\sin\theta + i\cos\theta$ A1

$= i(\cos\theta + i\sin\theta)$ A1

$= i\,y$ AG N0

OR

$i\,y = i(\cos\theta + i\sin\theta)\ (= i\cos\theta + i^2\sin\theta)$ A1

$= i\cos\theta - \sin\theta$ A1

$= \frac{dy}{d\theta}$ AG N0

(b) $\int \frac{dy}{y} = i\int d\theta$ M1A1

$\ln y = i\theta + c$ A1

Substituting (0, 1) $0 = 0 + c \Rightarrow c = 0$ A1

$\therefore \ln y = i\theta$ A1

$y = e^{i\theta}$ AG N0

(c) $\cos n\theta + i\sin n\theta = e^{in\theta}$ M1

$= (e^{i\theta})^n$ A1

$= (\cos\theta + i\sin\theta)^n$ AG N0

Note: *Accept this proof in reverse.*

(d) (i) $\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$ M1

Expanding rhs using the binomial theorem M1A1

$= \cos^6\theta + 6\cos^5\theta\, i\sin\theta + 15\cos^4\theta(i\sin\theta)^2 + 20\cos^3\theta(i\sin\theta)^3$
$+ 15\cos^2\theta(i\sin\theta)^4 + 6\cos\theta(i\sin\theta)^5 + (i\sin\theta)^6$

Equating imaginary parts (M1)

$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$ A1

$\frac{\sin 6\theta}{\sin\theta} = 6\cos^5\theta - 20\cos^3\theta(1-\cos^2\theta) + 6\cos\theta(1-\cos^2\theta)^2$ A1

$= 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$ ($a = 32$, $b = -32$, $c = 6$) A2 N0

(ii) $\lim_{\theta\to 0}\frac{\sin 6\theta}{\sin\theta} = \lim_{\theta\to 0}\left(32\cos^5\theta - 32\cos^3\theta + 6\cos\theta\right)$ M1

$= 32 - 32 + 6$

$= 6$ A1 N0

[20]

54. (a) $f'(x) = \ln x + x\left(\frac{1}{x}\right) - 1$ (M1)

$= \ln x$ A1 N2

(b) Using integration by parts

METHOD 1

$\int(\ln x)^2\,dx = (\ln x)^2 \bullet x - \int x \bullet \frac{2}{x}(\ln x)dx$ A1A1

$= x(\ln x)^2 - 2\int(\ln x)dx$ (A1)

$= x(\ln x)^2 - 2(x\ln x - x) + C$ A1

$(= x(\ln x)^2 - 2x\ln x + 2x + C)$

METHOD 2

$\int(\ln x)^2\,dx = x(\ln x)^2 - x\ln x - \int(\ln x - 1)dx$ A1A1A1

$= x(\ln x)^2 - x\ln x - (x\ln x - x - x) + C$ A1

$(= x(\ln x)^2 - 2x\ln x + 2x + C)$

Note: *Do not penalize the absence of + C.*

[6]

55. (a) x-intercepts are $x = 1$ and $x = \pi$ (accept 3.14) A1A1

(b) Attempting to find the area of two regions M1

$B = \int_{0.5}^{1}\sin x\ln x\,dx + \int_{\pi}^{3.5}\sin x\ln x\,dx$

$= -(0.09310... + 0.07736...)$

$B = 0.1704...$ (A1)

$A = \int_{1}^{\pi}\sin x\ln x\,dx = 0.8809...$ (A1)

$0.8809 = k \times 0.1704$

$k = 5.17$ A1 N2

Notes: *Accept values for A and B rounded to at least two decimal places.*
Accept only 5.17 for final A1.
Do not penalize if a negative value of B is used to yield a negative value of k.

[6]

56. $\frac{dy}{y} = \frac{x\,dx}{x^2+1}$ M1

$\int\frac{dy}{y} = \ln y$ A1

$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)$ A1

EITHER

$\ln y = \frac{1}{2}\ln(x^2+1) + \ln C$

$\ln y = \ln C\sqrt{x^2+1}$

$1 = C\sqrt{2}$ for substituting $x = 1$, $y = 1$ M1

$C = \frac{1}{\sqrt{2}}$ A1

$y = \sqrt{\frac{x^2+1}{2}}$ A1

OR

$\ln y = \frac{1}{2}\ln(x^2+1) + A$

$\ln 1 = \frac{1}{2}\ln 2 + A$ for substituting $x = 1$, $y = 1$ M1

$A = -\frac{1}{2}\ln 2$ A1

$\ln y = \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\ln 2 \left(= \frac{1}{2}\ln(x^2+1) - 0.347\right)$ A1

$\left(\ln y = \ln\sqrt{\frac{x^2+1}{2}}\right)$

$\left(y = \sqrt{\frac{x^2+1}{2}}\right)$

[6]

57. Separating variables

$\int \frac{dy}{y^2} = \int 2x\,dx$ A1

$-\frac{1}{y} = x^2 + C$ A1A1

Note: *The first A1 above is for a correct LHS and the second A1 is for a correct RHS that must include C.*

Using $y(0) = 1$ gives $C = -1$ $\left(-\frac{1}{y} = x^2 - 1\right)$ M1

$y = -\frac{1}{x^2-1}$ $\left(= \frac{1}{1-x^2}\right)$ A1 N0

[6]

58. $V = \pi\int_a^b y^2\,dx$ (M1)

$V = \pi\int_0^{\frac{\pi}{4}} \sin^2 3x\,dx$ A1

$\sin^2 3x = \frac{1}{2}(1-\cos 6x)$ (A1)

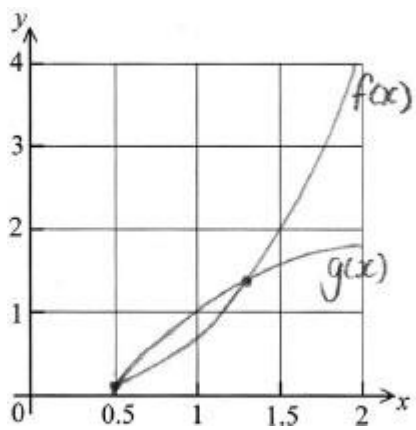
$V = \frac{\pi}{2}\int_0^{\frac{\pi}{4}}(1-\cos 6x)\mathrm{d}x$

$= \frac{\pi}{2}\left[x - \frac{1}{6}\sin 6x\right]_0^{\frac{\pi}{4}}$ A1

$= \frac{\pi}{2}\left(\frac{\pi}{4}+\frac{1}{6}\right)\ \left(=\frac{\pi^2}{8}+\frac{\pi}{12}\right)$ M1A1 N0

[6]

59. (a)



A1A1

(b) $A = \int_a^b g(x) - f(x)dx$ (using an appropriate definite integral) (A1)

$a = 0.50546...$, $b = 1.227...$ (A1)(A1)

$A = 0.201$ A1 N2

[6]

60. Let $u = \mathrm{e}^x$ M1

$\mathrm{d}u = \mathrm{e}^x\,\mathrm{d}x$ (or equivalent) A1

When $x = 0$, $u = 1$ **and** when $x = \ln 3$, $u = 3$ (A1)

$\int_0^{\ln 3}\frac{\mathrm{e}^x}{\mathrm{e}^{2x}+9}\mathrm{d}x = \int_1^3 \frac{1}{u^2+9}\mathrm{d}u$ A1

$= \frac{1}{3}\left[\arctan\frac{u}{3}\right]_1^3$ A1

$= \frac{1}{3}\left(\arctan 1 - \arctan\frac{1}{3}\right)\ \left(=\frac{\pi}{12}-\frac{1}{3}\arctan\frac{1}{3}, \frac{1}{3}\arctan\frac{1}{2}\right)$ A1 N0

[6]

61. (a) $\cos 2\theta = \cos(\theta + \theta)$ M1

$\cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$ $(\cos^2\theta - \sin^2\theta)$ A1

$= \cos^2\theta - (1-\cos^2\theta)$ A1

$= 2\cos^2\theta - 1$

$\therefore \cos^2\theta = \frac{\cos 2\theta + 1}{2}$ AG

(b) $\int \cos^2 x\,\mathrm{d}x = \frac{1}{2}\int(\cos 2x + 1)\mathrm{d}x$ A1

$\int \cos 2x\,dx = \frac{1}{2}\sin 2x$ (A1)

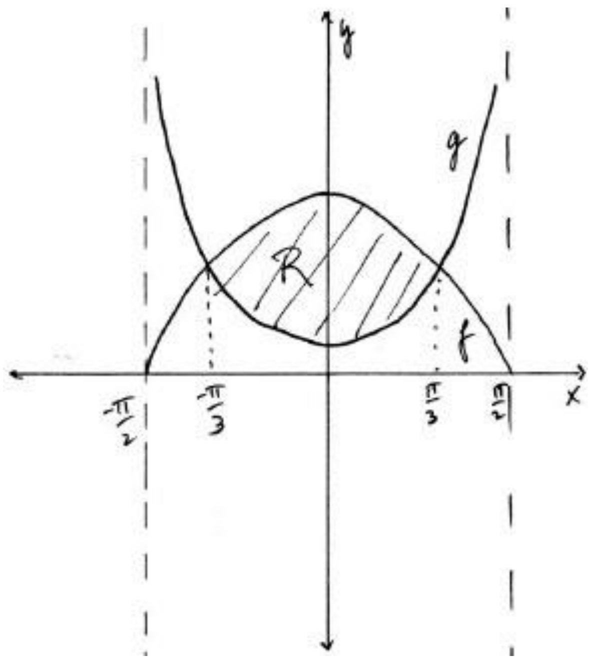
$\int \cos^2 x\,dx = \frac{1}{4}\sin 2x + \frac{1}{2}x + C$ A1A1

(c) Curves intersect when $f(x) = g(x)$ *ie* $4\cos x = \frac{1}{\cos x}$ (M1)

$\cos^2 x = \frac{1}{4}\left(\cos x = \pm\frac{1}{2}\right)$ A1

solving gives $x = \pm\frac{\pi}{3}$ A1A1 N3

(d)



A1A1A1

Note: *Award A1 for the basic shape of each graph and A1 for the shading.*

(e) (i) Using $V = \int_a^b \pi y^2\,dx$ (M1)

Volume = $\pi \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(16\cos^2 x - \sec^2 x\right) dx$ A1A1 N3

(ii) $16\int \cos^2 x\,dx = 16\left(\frac{1}{4}\sin 2x + \frac{1}{2}x\right)$ A1

$\int \sec^2 x\,dx = \tan x$ A1

Substituting limits

Volume = $2\pi\left(4\frac{\sqrt{3}}{2} + \frac{8\pi}{3} - \sqrt{3}\right)$ A1A1A1

$= 2\pi\left(\sqrt{3} + \frac{8\pi}{3}\right)$ A1 N0

[24]

62. (a) $I = \int \sin x \cos x\, e^{-\sin x}\,dx$

For a reasonable attempt at integration by parts. (M1)

$u = \sin x \qquad v = -e^{-\sin x}$

$du = \cos x\, dx \qquad dv = \cos x\, e^{-\sin x}\, dx$ (A1)

$I = -\sin x\, e^{-\sin x} + \int \cos x\, e^{-\sin x}\, dx$ A1A1

$= -\sin x\, e^{-\sin x} - e^{-\sin x} + C$ AG

(b) IF $= e^{\int -\cos x\, dx}$ (M1)(A1)

$= e^{-\sin x}$ A1

(c) $e^{-\sin x}\frac{dy}{dx} - y\cos x e^{-\sin x} = \sin x \cos x\, e^{-\sin x}$ M1A1

$e^{-\sin x} y = \int \sin x \cos x\, e^{-\sin x}\, dx$ M1A1

$e^{-\sin x} y = -\sin x\, e^{-\sin x} - e^{-\sin x} + C$ A1

Substituting $x = 0$ and $y = -2 \Rightarrow$ (M1)

$-2 = 0 - 1 + C$

$-1 = C$(A1)

so $e^{-\sin x} y = -\sin x\, e^{-\sin x} - e^{-\sin x} + -1$ A1

$y = -\sin x - 1 - e^{\sin x}$ A1

[16]

63. $V = \pi \int y^2\, dx$ M1

$= \pi \int_0^{2a} 8a(2a - x)dx$ A1A1

Note: *A1 for correct use of y^2, A1 for correct limits.*

$= 8\pi a\left[2ax - \frac{x^2}{2}\right]_0^{2a}$ M1

$= 8\pi a(4a^2 - 2a^2)$ (A1)

$= 16\pi a^3$ A1 N0

[6]

64. $\int_0^a \arcsin x\, dx = x \arcsin x_0^a - \int_0^a \frac{x}{\sqrt{1-x^2}}\, dx$ M1A1A1

$= a \arcsin a - 0 + \left[\sqrt{1-x^2}\right]_0^a$ A1A1

$= a \arcsin a + \sqrt{1-a^2} - 1$ A1

[6]

65. $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

$\Rightarrow \int \frac{1}{1+y^2} dy = \int \frac{1}{1+x^2} dx$ M1

$\Rightarrow \arctan y = \arctan x + k$ A1

$\Rightarrow \arctan \sqrt{3} = \arctan \frac{\sqrt{3}}{3} + k$

$\Rightarrow \frac{\pi}{3} = \frac{\pi}{6} + k \Rightarrow k = \frac{\pi}{6}$ A1

$\Rightarrow \arctan y = \arctan x + \dfrac{\pi}{6}$

$\Rightarrow y = \tan\left(\arctan x + \dfrac{\pi}{6}\right)$ M1

$\Rightarrow\ y = \dfrac{x + \tan\frac{\pi}{6}}{1 - x\tan\frac{\pi}{6}}$

$\Rightarrow\ y = \dfrac{x + \frac{\sqrt{3}}{3}}{1 - x\frac{\sqrt{3}}{3}}$ A1

$\Rightarrow\ y = \dfrac{3x + \sqrt{3}}{3 - x\sqrt{3}}$ A1 N0

[6]

66. $2 + x - x^2 = 2 - 3x + x^2$ M1

$\Rightarrow 2x^2 - 4x = 0$

$\Rightarrow 2x(x - 2) = 0$

$\Rightarrow x = 0, x = 2$ A1A1

Notes: *Accept graphical solution.*
Award M1 for correct graph and A1A1
for correctly labelled roots.

$\therefore \mathrm{A} = \int_0^2 \left(\left(2 + x - x^2\right) - \left(2 - 3x + x^2\right)\right)\mathrm{d}x$ (M1)

$= \int_0^2 \left(4x - 2x^2\right)\mathrm{d}x$ or equivalent A1

$= \left[2x^2 - \dfrac{2x^3}{3}\right]_0^2$ A1

$= \dfrac{8}{3}\left(= 2\dfrac{2}{3}\right)$ A1

[7]

67. **METHOD 1**

$V = \pi\int_1^e \left(\dfrac{\ln x}{x}\right)^2 \mathrm{d}x$ M1

Integrating by parts:

$u = (\ln x)^2, \dfrac{\mathrm{d}v}{\mathrm{d}x} = \dfrac{1}{x^2}$ (M1)

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{2\ln x}{x},\ v = -\dfrac{1}{x}$

$\Rightarrow V = \pi\left(-\dfrac{(\ln x)^2}{x} + 2\int \dfrac{\ln x}{x^2}\mathrm{d}x\right)$ A1

$u = \ln x,\ \dfrac{\mathrm{d}v}{\mathrm{d}x} = \dfrac{1}{x^2}$ (M1)

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{x}, v = -\dfrac{1}{x}$

$\therefore \int \frac{\ln x}{x^2} \mathrm{d}x = -\frac{\ln x}{x} + \int \frac{1}{x^2} \mathrm{d}x = -\frac{\ln x}{x} - \frac{1}{x}$ A1

$\therefore V = \pi \left[-\frac{(\ln x)^2}{x} + 2\left(-\frac{\ln x}{x} - \frac{1}{x} \right) \right]_1^{\mathrm{e}}$

$= 2\pi - \frac{5\pi}{\mathrm{e}}$ A1

METHOD 2

$V = \pi \int_1^e \left(\frac{\ln x}{x} \right)^2 \mathrm{d}x$ M1

Let $\ln x = u \Rightarrow x = \mathrm{e}^u, \frac{\mathrm{d}x}{x} = \mathrm{d}u$ (M1)

$\int \left(\frac{\ln x}{x} \right)^2 \mathrm{d}x = \int \frac{u^2}{\mathrm{e}^u} \mathrm{d}u = \int \mathrm{e}^{-u} u^2 \mathrm{d}u = -\mathrm{e}^{-u} u^2 + 2\int \mathrm{e}^{-u} u \,\mathrm{d}u$ A1

$= -\mathrm{e}^{-u} u^2 + 2\left(-\mathrm{e}^{-u} u + \int \mathrm{e}^{-u} \mathrm{d}u \right) = -\mathrm{e}^{-u} u^2 - 2\mathrm{e}^{-u} u - 2\mathrm{e}^{-u}$

$= -\mathrm{e}^{-u} \left(u^2 + 2u + 2 \right)$ A1

When $x = \mathrm{e}$, $u = 1$. When $x = 1$, u = 0.

$\therefore \text{Volume} = \pi \left[-\mathrm{e}^{-u} \left(u^2 + 2u + 2 \right) \right]_0^1$ M1

$= \pi \left(-5\mathrm{e}^{-1} + 2 \right) \left(= 2\pi - \frac{5\pi}{\mathrm{e}} \right)$ A1

[12]

68. Using integration by parts (M1)

$u = x, \frac{\mathrm{d}u}{\mathrm{d}x} = 1, \frac{\mathrm{d}v}{\mathrm{d}x} = \sin 2x \text{ and } v = -\frac{1}{2}\cos 2x$ (A1)

$\left[x\left(-\frac{1}{2}\cos 2x \right) \right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \left(-\frac{1}{2}\cos 2x \right) \mathrm{d}x$ A1

$= \left[x\left(-\frac{1}{2}\cos 2x \right) \right]_0^{\frac{\pi}{6}} + \left[\frac{1}{4}\sin 2x \right]_0^{\frac{\pi}{6}}$ A1

Note: *Award the A1A1 above if the limits are not included.*

$\left[x\left(-\frac{1}{2}\cos 2x \right) \right]_0^{\frac{\pi}{6}} = -\frac{\pi}{24}$ A1

$\left[\frac{1}{4}\sin 2x \right]_0^{\frac{\pi}{6}} = \frac{\sqrt{3}}{8}$ A1

$\int_0^{\frac{\pi}{6}} x \sin 2x \,\mathrm{d}x = \frac{\sqrt{3}}{8} - \frac{\pi}{24}$ AG N0

Note: *Allow FT on the last two A1 marks if the expressions are the negative of the correct ones.*

[6]

69. Let $u = \ln y \Rightarrow \mathrm{d}u = \dfrac{1}{y}\mathrm{d}y$ A1(A1)

$\displaystyle\int \frac{\tan(\ln y)}{y}\mathrm{d}y = \int \tan u \,\mathrm{d}u$ A1

$\displaystyle = \int \frac{\sin u}{\cos u}\mathrm{d}u = -\ln|\cos u| + c$ A1

EITHER

$\displaystyle\int \frac{\tan(\ln y)}{y}\mathrm{d}y = -\ln|\cos(\ln y)| + c$ A1A1

OR

$\displaystyle\int \frac{\tan(\ln y)}{y}\mathrm{d}y = \ln|\sec(\ln y)| + c$ A1A1

[6]

70. (a) Either solving $e^{-x} - x + 1 = 0$ for x, stating $e^{-x} - x + 1 = 0$, stating P(x, 0) or using an appropriate sketch graph. M1

$x = 1.28$ A1 N1

Note: *Accept P(1.28, 0).*

(b) Area = $\displaystyle\int_0^{1.278...}(e^{-x} - x + 1)\mathrm{d}x$ M1A1

$= 1.18$ A1 N1

Note: *Award M1A0A1 if the dx is absent.*

[5]

71. (a) (i) **EITHER**

Attempting to separate the variables (M1)

$\dfrac{\mathrm{d}v}{-v(1+v^2)} = \dfrac{\mathrm{d}t}{50}$ (A1)

OR

Inverting to obtain $\dfrac{\mathrm{d}t}{\mathrm{d}v}$ (M1)

$\dfrac{\mathrm{d}t}{\mathrm{d}v} = \dfrac{-50}{v(1+v^2)}$ (A1)

THEN

$\displaystyle t = -50\int_{10}^{5}\frac{1}{v(1+v^2)}\mathrm{d}v \quad \left(= 50\int_5^{10}\frac{1}{v(1+v^2)}\mathrm{d}v\right)$ A1 N3

(ii) $t = 0.732(\text{sec}) \quad \left(= 25\ln\dfrac{104}{101}(\text{sec})\right)$ A2 N2

(b) (i) $\dfrac{\mathrm{d}v}{\mathrm{d}t} = v\dfrac{\mathrm{d}v}{\mathrm{d}x}$ (M1)

Must see division by v ($v > 0$) A1

$\dfrac{\mathrm{d}v}{\mathrm{d}x} = \dfrac{-(1+v^2)}{50}$ AG N0

(ii) Either attempting to separate variables or inverting to obtain $\dfrac{\mathrm{d}x}{\mathrm{d}v}$ (M1)

$\displaystyle\int \frac{\mathrm{d}v}{1+v^2} = -\frac{1}{50}\int \mathrm{d}x$ (or equivalent) A1

Attempting to integrate both sides M1

arctan $v = -\frac{x}{50} + C$ A1A1

Note: *Award A1 for a correct LHS and A1 for a correct RHS that must include C.*

When $x = 0$, $v = 10$ and so $C = \arctan 10$ M1

$x = 50(\arctan 10 - \arctan v)$ A1 N1

(iii) Attempting to make arctan v the subject. M1

$\arctan v = \arctan 10 - \frac{x}{50}$ A1

$v = \tan\left(\arctan 10 - \frac{x}{50}\right)$ M1A1

Using tan $(A - B)$ formula to obtain the desired form. M1

$$v = \frac{10 - \tan\frac{x}{50}}{1 + 10\tan\frac{x}{50}}$$ AG N0

[19]

72. (a) $\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x}$ M1A1

$= 0$ AG

(b) Using integration by parts M1

$\int_0^a x e^{-x}\,dx = \left[-xe^{-x}\right]_0^a + \int_0^a e^{-x}\,dx$ A1A1

$= ae^{-a} - \left[e^{-x}\right]_0^a$ A1

$= 1 - ae^{-a} - e^{-a}$ A1

(c) Since e^{-a} and ae^{-a} are both convergent (to zero), the integral is convergent. R1

Its value is 1. A1

[9]